The Mathematics Teacher

MAY 1961

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H. E. WILLIAMS

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Printed at Menasha, Wisconsin, U.S.A. Second-class postage has been paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 412, P. L. & R., authorized March 1, 1930. Printed in U.S.A.

The Mathematics Teacher

volume LIV, *number* 5

May 1961

Learning by discovery, GERTRUDE HENDRIX 290
Finite planes and Latin squares, TRUMAN BOTTS 300
Square circles, FRANCIS SCHEID 307
If not solid geometry, then what? SEYMOUR SCHUSTER 313
The place of probability and statistics in the mathematics curriculum of the senior high school and junior college, H. E. WILLIAMS 316
Statistics for selected secondary-school students, CHARLES M. BRIDGES, JR. 321
An evaluation of a secondary-school mathematics program for able pupils, B. G. PAULEY 324
Some implications of programmed instruction for the teaching of mathematics, RALPH T. HEIMER 333
A Pascal pyramid for trinomial coefficients, EVELYN B. ROSENTHAL 336
Algebra and geometry; mathematics or science? ALBERT WILANSKY 339
Could we teach limits? LEHI T. SMITH 344
An application of the law of sines, EDWARD L. CLIFFORD 346
Introducing proof with a finite system, JOHN D. WISEMAN, JR. 351

DEPARTMENTS

Experimental programs, EUGENE D. NICHOLS
High school geometry via ruler-and-protractor axioms, MAX S. BELL 353
Historically speaking,—HOWARD EVES
Arithmetic problems 175 *years ago*, CECIL B. READ 361
The International Colloquium on Differential Geometry and Topology, LAURA GUGGENBUHL 363
A brief history of mathematics in rhyme, MARCELLA B. KRUEGER 365
Reviews and evaluations, KENNETH B. HENDERSON 368
Tips for beginners, JOSEPH N. PAYNE *and* WILLIAM C. LOWRY
Finding illustrative examples and exercises, DUNSTAN HAYDEN 373
Have you read? 312; *What's new?* 306
Letters to the editor, 335, 338

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Officers of the NCTM Affiliated Groups, EUGENE P. SMITH 378
Your professional dates 385
Twenty-first Summer Meeting 386
State representatives 387
More about 1960–1961 *Committees* 388

THE MATHEMATICS TEACHER is published monthly eight times a year, October through May. The individual subscription price of $5.00 ($1.50 to students) includes membership in the Council. For an additional $3.00 ($1.00 to students) the member may also receive *The Arithmetic Teacher*. Institutional subscription to either journal: $7.00 per year. Single copies: 85 cents each. Remittance should be made payable to *The National Council of Teachers of Mathematics*, 1201 Sixteenth Street, N.W., Washington 6, D.C. Add 25 cents for mailing to Canada, 50 cents for mailing to foreign countries.

*Learning by discovery**

GERTRUDE HENDRIX, *Teacher Co-ordinator, UICSM Project, University of Illinois, Urbana, Illinois.*

An analysis of some methods of teaching, considering the question "What is the discovery method of teaching?"

THE ISSUE OF LEARNING by discovery and of teaching for learning by discovery is beclouded at present by the fact that each of *three* very different procedures is being called *'The* Discovery Method.' They are the inductive method, the nonverbal awareness method, and the incidental method.

THE INDUCTIVE METHOD

Many people think that 'the discovery method' is just another name for the inductive method. The inductive method is nothing new in mathematics education. Colburn's book on teaching arithmetic by this kind of approach[1] was first published in the early 1820's. The companion volume on algebra[2] came a few years later. This method was strongly advocated in reports of the National Committee on Reorganization of Secondary School Mathematics in 1918 and 1923. That committee of the Mathematical Association of America drew heavily on earlier recommendations of E. H. Moore in the United States, Perry in England, and Felix Klein in Germany—mathematicians who were writing of these things around the turn of the century. Again in the early 1940's, the Joint Committee of the National Council of Teachers of Mathematics and the Mathematical Association of America, in a report published as a yearbook of the National Council,[3] strongly advocated the inductive method. These recommendations, however, were acted upon by such a small minority of teachers of mathematics that most people seeing the inductive method in action today think they are seeing something new. Furthermore, in clarity and productive outcome it is so far superior to an authoritarian "tell'm and drill" approach that, to someone whose school experience has been dominated by a formal approach, the inductive method seems the final answer to all pedagogical difficulties. Hence, someone just now beginning to think seriously about theory of instruction is likely to look upon the inductive method as the ultimate answer. *The fallacy in the inductive method lies in its confusion of verbalization of discovery with the advent of the discovery itself.*

The separation of discovery phenomena from the process of composing sentences which express those discoveries is the big new breakthrough in pedagogical theory. In some cases, correct verbalization of a discovery *does* emerge immediately after the dawning of awareness, but these cases are rare. They are confined to learners

* The writer acknowledges with gratitude many helpful suggestions from the following persons who read and criticized an earlier version of this article: Max Beberman, Alice G. Hart, Kenneth B. Henderson, R. Stewart Jones, David A. Page, and J. Richard Suchman, University of Illinois; Merrill B. Hill, Utah State Department of Public Instruction; and Stanley M. Jencks, University of Utah.

[1] Warren Colburn, *First Lessons; Intellectual Arithmetic upon the Inductive Method of Instruction*, new edition, with an introduction to written arithmetic by his son, Warren Colburn, and an introduction by George B. Emerson (Boston: Hilliard, Gray, Little and Wilkins, 1863). [Original edition about forty years earlier; others in 1849, 1858.]

[2] Warren Colburn, *An Introduction to Algebra upon the Inductive Method of Instruction* (Boston: Cummings, Hilliard, and Company, 1826).

[3] National Council of Teachers of Mathematics, *Fifteenth Yearbook*, (New York. Bureau of Publications, Teachers College, Columbia University, 1940). Cf. item (c) p. 57.

with unusual powers of correctness and precision in the use of a mother tongue, and they are confined to situations in which the learner happens to possess all the vocabulary and rules of sentence structure needed to formulate the new discovery. In most cases, for evidence that an inductive discovery has occurred, a teacher must be able to recognize responses or behavior other than a linguistic formulation of the discovery. The widespread confusion of sentence formation with the discovery process itself has been responsible for much of the frustration, discouragement, and ultimate abandonment of the inductive method on the part of many teachers who have attempted to master it. As one head of a teachers college department of education has put it, "But the only ones who can state a generalization from examining examples are the same exceptionally bright ones who could have learned to apply the generalization from studying someone else's statement of it in the first place!" As a matter of fact, the only teachers who have been really successful with the inductive method are the rare ones who have acquired the knack of making the learner *aware* of generalizations before calling for verbalizations. They have been doing "right things for wrong reasons," and the subsequent untimely verbalizations have only partially marred the outcomes.

If the writer may be permitted so personal a statement, I should like to record here that it was while in pursuit of greater skill with the inductive method that I first became conscious of the nonverbal awareness stage in discovery learning. From concentrated work in helping student teachers to acquire skill in this method, I had become very conscious of three bad mistakes which beginners are likely to make when they plan and attempt inductive teaching:

1. They begin to call for generalizations, exerting anxious pressure on their classes, long before the students have noticed any basic similarity among the examples the teacher has presented. The discussion degenerates into a guessing contest during which the students try to find out "what it is the teacher wants me to say."
2. The teacher often calls for statements of generalizations when the students do not possess the vocabulary and rules of sentence formation necessary for a precise verbalization of the generalization, even if they have "seen it."
3. The teacher often confuses generalizations, which had to be discovered in the first place and, hence, are appropriate subject matter for rediscovery, with situations in which the generalization is arbitrary—something that is merely a matter of definition.

A frequent costly consequence of item 2 above is a teacher's yielding to a temptation to accept a faulty answer, an answer which is either false or nonsense. Spontaneous verbal responses at moments of discovery are usually expressive of emotion only. Often they are ejaculations, referentially haphazard. Sometimes they are in sentence form but seem to be utter nonsense if repeated and examined. Or, still worse, they may be plausible, *interpretable* sentences which state false generalizations. Spontaneous verbal fragments often provide clues from which a clever teacher can infer what is in a student's mind, and for that purpose they are valuable. But, unless the teacher is a tutor working with only one pupil, encouraging such responses can be very damaging to the learning under way in *other* members of the class. Even in the tutoring situation, an incorrect spontaneous verbalization of a discovery—a sentence which really says something else—must be ignored and forgotten as soon as possible. Accepting it as a "nice try" can cause an involuntary interpretation of the false sentence to prevail over the correct discovery. If *any* recognition is accorded to such a response, the teacher must stick with it until counter examples have revealed to the whole class that the sentence must be discarded. By this time the class has shifted to an ex-

ercise in language composition, which cannot be abandoned tactfully. Pushing on through to a correct verbalization at such a time usually demands a long, laborious digression. By the time the point is clear, most of the zest for the original learning has evaporated; and the production of the correct sentence, accepted at long last, may even destroy the anticipatory attitude with which ensuing applications of the generalization should be approached. (Experimentation has established that this effect of finality and detachment is avoided if verbalization of a generalization is postponed to a later lesson. At that time the linguistic formulation of things already "known" can be undertaken as an end in itself.)

The big problem in the inductive method has been to find some way of telling when students are far enough along in an inductive approach for it to be *fair* to ask them to state a generalization. It was while a search was under way for a criterion by which to identify this stage in each inductive lesson that an event came along which led to my identification of the nonverbal awareness stage in discoveries of concepts and generalizations. This happened in 1937.

As early as 1946, clinical experts in nondirective psychological counseling were pointing out that frequently there were very significant changes in behavior based on unverbalized insights, and that the effectiveness of an interview could be destroyed by premature pressure for verbalization. The counseling experts were *not* telling us, however, how to detect when it was *not too soon* to press for verbalizations. When I designed my first experiment[4] for research on this question, for one group of subjects the teaching was terminated at the nonverbal awareness stage simply to call attention to the fact that that stage existed, that there *was* a way to tell when a learner was aware of a generalization which he might be asked to state. At the time that I designed this part of the experiment, it did not occur to me that as far as transfer power was concerned, the *whole* thing was there as soon as the nonverbal awareness had dawned. No one was more shocked than I was to find that not only did the learners who completed correct verbalization of the discovery do no better on transfer tests than those for whom the teaching was terminated at the nonverbal awareness stage, but also that the verbalization of the discovery seemed actually to have diminished the power of some persons to apply the generalization. Charles Hubbard Judd in his conscious generalization theory of transfer had considered 'generalizing' as synonymous with 'composing a sentence which states the generalization involved'.[5] But that simply is not the case; unless Judd's definition of 'generalizing' is changed, we can no longer say that generalizing is the primary generator of transfer power.

Since 1946, the problems have become:

a) How can teaching be planned and executed so that necessary verbalizations of discoveries are accomplished *without* damage to the dynamic quality of the learning itself?
b) What is there about the nature of language and the process of acquiring a language, and what goes on when one uses language that could explain (and, hence, help us to avoid) detrimental effects of verbalization?

Very soon a tremendously important by-product of this search began to appear: The discovery process itself is so exhilarating (to both children and adults) that it becomes its own motive in academic work. As a solution to the motivation problem for children in a free society whose existence is threatened by rapidly increasing

[4] Gertrude Hendrix, "A New Clue to Transfer of Training," *Elementary School Journal*, XLVIII (December 1947), 197–208.

[5] "The human power of generalization is so intimately related to the evolution of language that the two cannot be thought of as existing separately . . . language is the chief instrument of generalization. . . ." C. H. Judd, *The Psychology of Secondary Education*, (Boston: Ginn and Co., 1927).

scientific and technical power in the land of an enemy, the promotion of successful learning by discovery has no parallel. But such teaching is an art with a difficult technique. Furthermore, it is dependent upon sound and clearly written textbook materials. Ambiguities, inconsistencies, "fuzzy" writing tolerated for centuries—all these must be eliminated from science and mathematics courses. (Another field in which such stumbling blocks abound is that of grammatical construction of word languages. The unifying concepts common to all languages are now revealed by developments of the last half century in symbolic logic, but as yet these developments have not been utilized in a reorganization of grammar.)

The nonverbal awareness method

This method has been described incidentally in the above discussion of the inductive method. It will be further explained, perhaps, in the illustration developed in the second part of this article. This is the method of teaching which has been found most successful with the UICSM Mathematics Project materials. The materials are written to promote use of the method with a minimum of difficulty for a teacher to whom both materials and method are new.

The incidental method

This is an approach widely promoted during the Progressive Education era.

It is the method in which the school sets the stage for many experiences, which are usually being built around some central problem or project. Those who advocated this method knew that only generalizations which emerged from experiences were likely to play a dynamic part in the learner's later behavior and problem-solving activity. But all too often they took no responsibility for seeing that instances of the same generalization came along close enough together for the learner to become aware of either concepts or principles. With no underlying explanatory theory, those educators simply did not realize that most learners under such conditions seldom "broke through" to awareness of abstractions. Such a school program was doomed to triviality, except in the hands of a teacher who did many additional things, even though he did them for unsound reasons. Many persons today associate this triviality of results, from many of the so-called 'activity programs' with learning by discovery. They thus reject anything called 'discovery method'.

Illustrations

The three separate and very different teaching procedures, each referred to by its proponents as *'the* discovery method,' can be illustrated by reference to ways of teaching a multiplication rule for quotients. All three of the so-called discovery techniques are in contrast to any approach through initial statement of a rule. Just to get that approach out of our minds let's examine two possible statements of such a rule, one verbal, the other mathematical:

1. For each first number, for each second number not zero, for each third number, for each fourth number not zero, the quotient of the first number by the second number multiplied by the quotient of the third number by the fourth number is the product of the first number by the third number divided by the product of the second number by the fourth number.

2. $\forall_a \forall_{b\neq 0} \forall_c \forall_{d\neq 0} \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

Such sentences are avoided in traditional textbook approaches by resorting to garbled mixtures of arithmetic and metamathematics—rules which abandon references to numbers and tell one how to manipulate figures to get an answer. The very existence of such language as '. . . the product of the numerators over the product of the denominators' traces to confusion between numerals and the entities which numerals represent. Mathematicians have become aware of entities and

relations between the entities; they have then named, or symbolized, the entities and described the relations between them; and then they have thought that they were giving us the essence of their discoveries when they gave us the symbols and sentences. In the resulting desperation to make things clearer, they have composed verbal descriptions of *their symbols and what they do with them*, a sorry substitute for making those with whom they are trying to communicate aware of the concepts and generalizations they themselves have in mind on a nonverbal level. Stating an arithmetic generalization concerning products of quotients requires sentences about numbers. If a mathematician says what he means, he needs sentences of the types illustrated in (1) and (2). In contrast, let us see how discovery lessons based on each of the three interpretations of 'discovery method' outlined above would proceed.

Prerequisite to all three of the approaches are these previous learnings:

1. When a rectangle is divided into six congruent squares, each square is called 'one-sixth of the whole'; or, in general, for each counting number n, if a rectangle is divided into n congruent squares, then each square is called 'one-*nth* of the whole rectangle'.
2. When a rectangle two units wide and three units long is divided into unit squares, the resulting six squares can be thought of as two of three, or as three of two, that is, two rows of three squares each, or three columns of two squares each. That is, for each counting number m, for each counting number n, a rectangle m units long and n units wide is made up of m of n squares or n of m squares.
3. The number of unit squares in any rectangle of the kind described above is mn, that is, multiplication is the arithmetic process by which one finds the area-measure of a rectangle when he knows its length-measure and width-measure.

Inductive method

The class reviews the three items above, probably in the reverse order from that in which they are stated above. [It would be a more natural quasi-research approach if a problem to be solved had been stated first, for example:

What is $\frac{1}{2}$ of $\frac{3}{4}$?

Unfortunately, things are not usually done in this order. Hence, the prerequisite review is imposed authoritatively by the teacher, instead of being something thought of as a way of shedding light on a new, unsolved problem. Conducting a review before stating the problem injects a false and unnatural element into the discovery process, and robs the children of experience in an important technique of research.]

Second, if not earlier, the problem is stated, and the teacher helps the student to find an answer to the question, "What is $\frac{1}{2}$ of $\frac{3}{4}$?" by making appropriate drawings (Figures 1, 2, 3) and then counting.

First, a rectangle is divided into fourths and three of them are shaded. Next, the shaded part is divided into halves, and cross shading is used to identify one-half

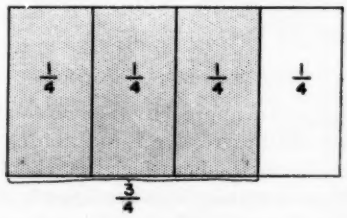


Figure 1

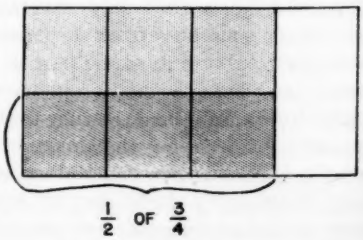


Figure 2

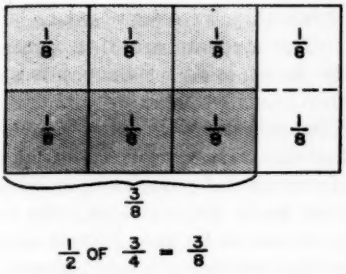


Figure 3

of three-fourths of the original rectangle. Finally, each part of the cross-shaded section is found by counting to be one-eighth of the original rectangle. Then, by counting, the children see that one-half of three-fourths of the rectangle is three-eighths of the rectangle. So, the teacher records on the chalkboard at one side, possibly under the heading 'Data':

$$\frac{1}{2} \quad \text{of} \quad \frac{3}{4}=\frac{3}{8}\,.$$

Referring back to 'two of three' and 'three of two' as synonyms for '3×2' and '2×3,' all agree that it should be all right[6] to restate the last problem as:

$$\frac{3}{4}\times\frac{1}{2}=\frac{3}{8}\,.$$

The period at the end of the first line of data is erased and the line is changed to this restatement of the problem:

$$\frac{1}{2} \quad \text{of} \quad \frac{3}{4}=\frac{3}{8}, \qquad \text{or } \frac{3}{4}\times\frac{1}{2}=\frac{3}{8}\,.$$

This process of making and examining appropriate drawings is repeated for at least three more examples, but usually not more than that. By this time, the recorded data may look something like this:

$$\frac{1}{2} \quad \text{of} \quad \frac{3}{4}=\frac{3}{8}, \qquad \text{or } \frac{3}{4}\times\frac{1}{2}=\frac{3}{8}\,.$$

$$\frac{2}{3} \quad \text{of} \quad \frac{1}{5}=\frac{2}{15}, \qquad \text{or } \frac{1}{5}\times\frac{2}{3}=\frac{2}{15}\,.$$

[6] In the UICSM materials the "times" sign, '×,' means *multiplied by*. That is, in '$\frac{3}{4}\times\frac{1}{2}$,' the number named second is the multiplier.

$$\frac{3}{4} \quad \text{of} \quad \frac{3}{5}=\frac{9}{20}, \qquad \text{or } \frac{3}{5}\times\frac{3}{4}=\frac{9}{20}\,.$$

$$\frac{3}{7} \quad \text{of} \quad \frac{2}{3}=\frac{6}{21}, \qquad \text{or } \frac{2}{3}\times\frac{3}{7}=\frac{6}{21}\,.$$

A teacher alert to the structural properties of the set of numbers of arithmetic will also have, for example:

$$\frac{3}{4} \quad \text{of} \quad \frac{1}{2}=\frac{3}{8}, \qquad \text{or } \frac{1}{2}\times\frac{3}{4}=\frac{3}{8}\,.$$

The differences between the experimental procedure for this exercise and the drawings in Figures 1 to 3 will have been illustrated on the board as follows with Figures 4 to 6:

$$\text{New Problem: } \frac{3}{4} \quad \text{of} \quad \frac{1}{2}=?$$

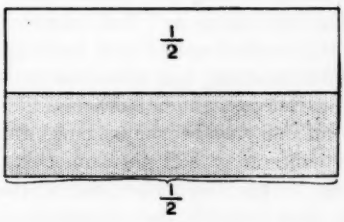


Figure 4

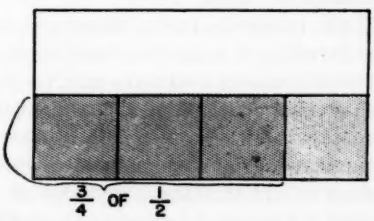


Figure 5

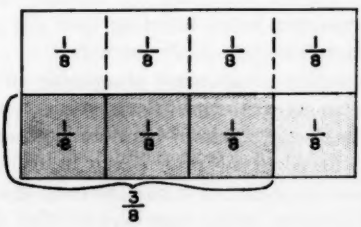


Figure 6

So $\frac{3}{4}$ of $\frac{1}{2}=\frac{3}{8}$, or $\frac{1}{2}\times\frac{3}{4}=\frac{3}{8}$, and a corresponding entry is included in the table of data.

The next part of the inductive lesson is usually introduced by the teacher with some remark, such as "Now, let's see how we could have found the answers just by looking at the fractions." The children are told to examine each pair of fractions and the fraction which names the product, and to try to tell how they could have found the product without drawing the picture. In the inductive method, *the ability to "tell how to do it" is the criterion by which the teacher recognizes that discovery has taken place.*

As a matter of fact, children at this stage in arithmetic do not have the linguistic machinery for stating precisely what they mean when they *have* discovered the rule. The inductive method forces them into saying such things as "Well, you multiply the top numbers, and you multiply the bottom numbers, and then you write the top answer over the bottom answer."

A teacher's insistence on more polished statements may call for something like "Well, to get the numerator of the answer, you multiply the numerator of the first fraction by the numerator of the second fraction; then. . . . "

If the teacher is one of those rare ones who is trying to keep clear the distinction between numbers and numerals, he must then go back and remind the student that a fraction is not a number; it is a *name* for a number. So the student must revise his answer to say that the numerator of the answer is a numeral for the product of the numbers named by the numerators of the two original fractions, etc. All this time, the teacher keeps insisting that "If you can't tell it, you don't know it."

Finally, a statement acceptable to the teacher is achieved. Each student may write the new rule in his notebook, or he may be asked to turn the page in his book, where he will find a rule already stated. (Of course, some members of the class may have looked ahead already and found out what they were supposed to be saying.) The students are then considered ready to work a set of practice exercises.

When well done, this method does give each student what is sometimes called 'an intuitive crutch,' a device upon which he can fall back; he can repeat the experiment whenever he finds himself in doubt about the rule. Security and independence are provided, and, in contrast to a tell-them-and-drill approach, the inductive lesson seems a revelation in pedagogy. A well-presented authoritarian approach is very satisfying to someone who *is already aware of* the ideas being presented; but for a learner to whom the ideas are new, the authoritarian method at its best is a *feeble* best in comparison with the inductive method. And *the inductive method at its best* is a feeble best in comparison with a genuine discovery approach which promotes and recognizes a nonverbal awareness stage in creative thought.

The nonverbal awareness method

Let us contrast the procedure called the inductive method with a discovery lesson which follows more closely a pattern of genuine research. The goal, as in the preceding development, is the "multiplying fractions" rule.

In this approach, statement of a problem *must* precede the developmental review leading to the physical space drawing experiment by which an answer is found. From then on, the lesson may proceed exactly as in the inductive method up to a certain point. Then a highly significant, sharply defined difference appears. In *this* kind of discovery approach, as soon as each child has mastered the drawing-and-counting device for finding products of pairs of quotients, he is given a list of problems to work. As far as the student knows, he is to work every problem on the list by making an appropriate drawing. The class activity thus shifts to individual seat work. What is not done before the bell

rings is to be finished for homework. The problems on the list should be arranged to promote interesting observations. The work sheet should provide a space for recording the answers to the problems. (At least two pairs of consecutive examples will illustrate the commutative principle for multiplication.) To give pause to the conscientious little task-doer who might work the whole list by making drawings just because that is what the directions (or the teacher) said to do, examples like the one below should be included:

$$\frac{17}{21} \times \frac{11}{19} = ?$$

Students accustomed to this kind of discovery approach will, of course, be on the lookout for a short cut. Furthermore, from previous experience with the course and their teacher, pupils will know that finding and using a short cut is something that will be highly approved. Until this experience has been built up, the directions for the exercise must contain some challenge, such as "If at any time you think you know what an answer is going to be *before* you make the drawing, please write down the answer and raise your hand." When a hand goes up, the teacher should go to the student, congratulate him, or advise him to check his hunch by another experimental drawing. It must be pointed out, however, that this direction injects a false note into the stage setting for true research. It is simply a device by which a teacher helps a class to get its first experiences with this kind of learning.

In such a lesson, *how does the teacher know when a discovery has taken place?* Suppose he is walking about the class looking over shoulders. A sudden start, a flush of excitement, and a student begins writing answers almost as fast as he can put them down. When a student begins to write correct answers without making the drawings, we have evidence of the advent of awareness. Awareness of what? Awareness of the "multiplying fractions" rule.

If the child were not aware of it, he couldn't be using it. To call a halt in his progress at this point for a long, tedious verbalization attempt is damaging in several ways. In the first place, it belittles the exhilarating accomplishment of the actual discovery. Whether he can "tell it" or not, the *child* knows very well that he *has* something.

Then, in the second place, the verbalization attempt at this grade-school level is doomed to frustration. Numerical variables, universal quantifiers, and the zero restrictions on domains are all necessary to a precise formulation of the generalization the child has just discovered. The most that a teacher can do is to accept a garbled verbalization of what was at the beginning a clear, dynamic insight. The more sensitive a child is to precision in the use of his mother tongue, the more damaging this process can be to him and to the learning which a moment before was so clear on the nonverbal awareness level. The more skillful the child is in interpreting his mother tongue, the higher the probability that a literal interpretation of his own incorrect statement will impinge itself upon him. If this literal interpretation of his incorrect statement contradicts the awareness that was already there, the original learning is mutilated.[7] In a lesson of this kind, the child who notices that his neighbor is writing correct answers without making drawings is electrified into a search for "what's going on." Knowing that one of his classmates has discovered a short cut has a much more highly motivating effect upon him than a search for short cuts suggested by the teacher. The child who does not discover the rule before the end of the class period still has his chance during his homework period—if well-meaning parents do not step in and spoil it all for him!

[7] I use the word 'mutiliated' deliberately and almost angrily. My feeling about this issue has been built up from several years' experience in seeing this thing happen to children over and over again.—The Author

On the following day the teacher presents other examples, such as:

$$\frac{5}{3}\times\frac{7}{8}$$

$$\frac{6}{5}\times\frac{9}{4}$$

$$\frac{6}{2}\times\frac{21}{7}.$$

In other words, *over*generalization and incomplete generalization are prevented by the introduction of *examples* which correct possible false impressions, *not* by verbalizing the rule at this stage of the game and conducting a sophisticated search for false instances. (The more sophisticated procedure is available and appropriate a few years later when these same children will have acquired the necessary linguistic equipment.)

I do not wish to belittle the tremendous improvement in instruction which results from pursuing the inductive method, in contrast to authoritarian procedures. But those who think teaching for discovery *is* the inductive method are still blinded to this second tremendous breakthrough in theory of instruction. It is recognition of the nonverbal awareness stage in inductive learning that converts the classroom experience into that of actual discovery, the kind of thing that promotes a taste for and a delight in research.

The incidental method

In contrast to both of the instructional procedures outlined above, let us think about the teacher who trusts that the "multiplying fractions" rule will be discovered incidentally.

In one project a child will need to find the cost of a sheet of gold leaf to line a jewelry box, for instance, when he knows the cost per square foot and knows that the sheet must be one-half foot wide and three-fourths foot long. He is taken through the drawing experiment until he finds an answer. No other instances of multiplying one quotient by another come along in *that* project. Six weeks later (or six months, or a year) in connection with another project, he has another occasion to find the product of two quotients. Again the physical space drawings are used, the answer is found, and the project proceeds. All pressure from teacher and group is directed toward finishing the project. (Anyone who becomes intrigued by the multiplication process itself and digresses to investigate will be frowned upon by many of the educators advocating this kind of schoolwork.) From such examples widely distributed over time and space, the probability that the generalization will emerge is almost zero. Only a little genius who has somehow escaped with his taste for contemplation undestroyed will recall such widely scattered experiences all at the same time, and hence become aware of the mathematical generalization underlying them. Critics of education who confuse this kind of school program with everything else referred to as 'learning by discovery' are justified in opposing a discovery method. Much as I, the writer, abhor authoritarian presentations of sentences which express generalizations, the words *later* to be clarified by examples, I would prefer falling back upon that relatively deadly and deadening procedure, rather than to rely upon important basic principles "coming clear" from experiences incidental to so-called 'project activity'.

So much for the status of the discovery method in educational parlance and educational literature at the beginning of the decade to be known as 'the sixties'! It is high time, however, for another category to be added to the list when discovery in *mathematics* is under discussion. Even the instructional procedure described under the nonverbal awareness method is not properly called '*the* discovery method'. It *is* possible to arrive at discovery by pure deduction, that is, by manipulating sentences according to logical rules. It is not likely, but neither is it impossible, that the

"multiplying fractions" rule would be discovered by derivation; most deductive proofs are checks for sentences first arrived at inductively; but *some* very important principles, both in mathematics and in physical science, *have* been discovered by deduction. A good example of something in secondary-school mathematics usually discovered (that is, *re*discovered) in this way is the rule for finding the measure of one side of a triangle in terms of measures of the other two sides and the cosine of the measure of their included angle. This is a case in which one begins with some sentences which express observations of a drawing. Starting with these sentences as premises, pure deduction takes over and out comes the desired formula. This last sentence, discovered deductively, is then interpreted in terms of how the variables were used in the original drawing. This interpretation of a final sentence in the deductive sequence reveals the desired geometric rule. This is a discovery process just as truly as is the inductive leap that takes place in the nonverbal awareness type of discovery. How do we know that both are discovery? Because a person can do them *all by himself*. The thing learned is not received by communication.

Much of the confusion permeating discussions of learning by discovery comes from failure to distinguish discovery from communication. One does not *discover* a concept or a principle in a book. He may discover there a *name* of a concept, a definition, a *sentence* which states a principle, etc., but before he "has" anything, he must engage himself in the process of interpreting the words and sentences of the author, that is, he must complete a communication process. Communication[8] usually plays an important part in setting the stage for discovery in the teaching of mathematics, but that part is quite different from actually communicating the new thing to be learned.

[8] A rationale of communication adequate for explaining this distinction has emerged from the nonverbal awareness research at the University of Illinois; the author hopes to make the theory available in print soon.

Experiments in mathematics

The Board of Directors of the National Council of Teachers of Mathematics appointed a committee to gather information and to analyze experimental mathematics programs. This committee would like your help in identifying such programs. It has limited its analysis to programs that (1) have printed instructional materials other than courses of study, (2) are of at least one semester in duration, and (3) are not sponsored by a commercial publishing firm. The committee will examine all material, ask the experimenter to make a short personal report, and publish as many of the reports as possible.

If you know of experimental programs or have one of your own, we would appreciate receiving your material or having you contact the chairman, whose address is:

PHILIP PEAK, *Chairman*
Committee on the Analysis of
Experimental Programs
School of Education
Indiana University
Bloomington, Indiana

*Finite planes and Latin squares**

TRUMAN BOTTS, *University of Virginia, Charlottesville, Virginia.*

New mathematical facts are constantly being discovered, and new mathematical ideas are continually being evolved. This is illustrated by developments in finite geometry.

ALMOST EVERYONE KNOWS that physics and chemistry are sciences in which new discoveries are made very day; but there are still people who think of mathematics as a "dead" or inactive field, in which all the important ideas were discovered hundreds of years ago. It comes as a great surprise to such people to find that new facts are constantly being discovered in mathematics, just as they are in physics or chemistry.

MATHEMATICS IN PROGRESS

If you would like to get some idea of the bewildering rate at which mathematical discoveries are being made today, and at the same time find out at least the *names* of the fields within mathematics in which these discoveries are being made, try thumbing through some current issue of *Mathematical Reviews*, a magazine which can be found, if not in your school library, certainly in any university library. *Mathematical Reviews* is what is called an abstract journal. It comes out once a month, and each issue contains short descriptions or abstracts of mathematical discoveries that have been published within the past year or so in research journals the world over. Each monthly issue of *Mathematical Reviews* gives abstracts of some seven or eight hundred research papers. In this country alone there are currently published about thirty journals devoted to such mathematics research papers. So you can see that mathematical discovery is a human activity which is very much alive today.

I'd like to illustrate, as well as I can in a short time, this continuing process of mathematical discovery by telling you a little about just one example. This particular example comes from the general field of geometry. You may not know it, but there are many kinds of geometry. The Euclidean plane geometry that most of you are already studying is just one, though a very important one, of these kinds of geometry. Let us go back to it for a moment.

FINITE GEOMETRY

As you know, in ordinary Euclidean plane geometry we start by supposing there are given some things, called points, and certain sets or collections of points, called lines such that certain assumptions or axioms are satisfied. We may have some intuitive ideas as to how points and lines should be pictured on the blackboard, but all the theorems in plane geometry are derived ultimately from these assumptions, or axioms, not from pictures. (Of course, there's nothing wrong with drawing pictures to get *suggestions* as to theorems that might be true and ways that they might be proved!)

In geometry, then, what theorems we get will depend on what assumptions or axioms we adopt. To get the "right" theorems (that is, the intuitively expected ones), mathematicians had to choose the axioms of Euclidean geometry pretty carefully. As a matter of history, we now know that Euclid didn't choose them quite

* This address was delivered under the Visiting Lecturer Program to Secondary Schools, sponsored by the Mathematical Association of America with the support of the National Science Foundation.

carefully enough, and that a really satisfactory set of axioms for ordinary Euclidean geometry was only obtained about sixty-five years ago, by the great German mathematician David Hilbert.

The kind of geometry I want to tell you a little about is called *finite projective plane geometry*. It has a much simpler set of axioms than ordinary Euclidean plane geometry has. By definition, a projective plane consists of some things called points and certain sets or collections of these points called lines, such that:

AXIOM 1: Every two points are contained together in exactly one line.

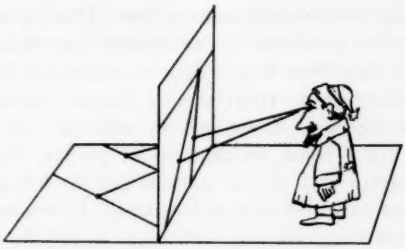
AXIOM 2: Every two lines contain in common exactly one point.

AXIOM 3: Every line contains at least three points.

AXIOM 4: There are at least two lines.

These are all the axioms! Let us look at them for a moment. The first one we can easily understand. It just says in slightly more careful language that two points determine a line. So far this is just like ordinary Euclidean geometry. The second axiom says that two lines always intersect in a point. This is different from Euclidean geometry. In ordinary Euclidean plane geometry two lines need not intersect in a point—they may be parallel instead. In projective plane geometry, then, there are no parallel lines!

This might seem to be a nonsensical sort of axiom, but I think I can make it seem a little more understandable and, at the same time, account for the name *projective* geometry. When an artist draws a realistic flat landscape, he tries to imagine it *projected* onto his canvas. That is, thinking of his canvas as a transparent pane of glass through which he is viewing the landscape, he tries to represent each point of the actual landscape by that point on the glass pane which lies directly in line with the actual point on the landscape and his own eye. Historically, it was the theory of such projective drawing that led to the study of projective geometry in the first place.



How does the artist draw the horizon line on his canvas? He draws it so that lines from his eye through it are essentially parallel to the plane of the landscape itself. How does he draw parallel lines, like the rails of a straight railroad, which extend to the horizon in the plane of the landscape? He draws them as lines which *do intersect in a point*, on the horizon line of the drawing.

Returning to our axioms, Axioms 3 and 4 are certainly not surprising since they are obviously satisfied in ordinary Euclidean plane geometry. As a matter of fact, in the Euclidean plane geometry that you study there are *infinitely many* points on every line and *infinitely many* lines. For a projective plane, though, there seems to be at least the possibility that there might be only finitely many points on some lines. Actually, it turns out that there are projective plane geometries with only a finite number of points in all! Such geometries are called *finite projective geometries*. They were first studied systematically in the year 1906 by the American mathematicians Oswald Veblen and W. H. Bussey.

In a moment I'll show you an example of a finite plane geometry. But first let's pause and examine this idea that a plane or a line might reasonably consist of a finite number of points. Intuitively we picture a line as being somehow like a tightly stretched string or a straight chalk mark on the blackboard. Starting from these intuitive ideas, how could we ever be led to consider a set of three or four points as constituting a line? Well, suppose you and I are trying to formulate very carefully the axioms we think the mathematical notion of line should satisfy, in order to fit

our intuitive ideas about lines. Then, after we've listed all the axioms we can think of that lines should satisfy, suppose some clever but troublesome person comes along and shows us an example of a "line," some peculiar set of points, that agrees with all our axioms but doesn't fit our intuitive idea of line at all. Of course, one thing we can then do is to adjoin or include additional axioms, to rule out the troublesome example and thus bring our mathematical axioms about lines into closer accord with our intuitive ideas. In the history of mathematics this sort of thing has often been done. But sometimes something else also happens: the peculiar "lines" that the troublesome person thought up turn out to be interesting in themselves and to have unsuspected connections with entirely different branches of mathematics! This is just what has happened with the lines of finite projective planes, as we shall see. We continue to call the new objects "lines," because they do have *some* of the properties we intuitively associate with lines; and thinking of them as lines is useful because it turns out to suggest other properties that we are able to prove about them.

An example

It is easy to show you a finite projective plane geometry. Consider (in the ordinary Euclidean plane) a circle with a circumscribed equilateral triangle with its altitudes, as shown in Figure 1.

Let us agree to consider only the seven numbered points shown in Figure 1 as being points of our projective plane. Let us also agree that the lines of the projective plane will be just those triples of numbered points which lie together on a side or altitude of the triangle or on the circle. The "drawn lines" on the paper are present to help group the points together into lines and aid the eye. The lines consist of only three points. It is easy to check that all four axioms are satisfied, so even though this geometry may seem rather strange, it has to be accepted as legitimate.

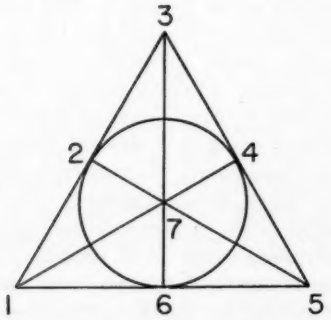


Figure 1

If we list the lines, we find that these are seven in number. Thus, we can label the rows in a 7 by 7 square with these lines and the columns in the same square with the seven points of the geometry, as shown in Figure 2.

Incidence matrix

Lines \ Points	1	2	3	4	5	6	7
(1, 2, 3)	1	1	1	0	0	0	0
(3, 4, 5)	0	0	1	1	1	0	0
(1, 5, 6)	1	0	0	0	1	1	0
(1, 4, 7)	1	0	0	1	0	0	1
(3, 6, 7)	0	0	1	0	0	1	1
(2, 5, 7)	0	1	0	0	1	0	1
(2, 4, 6)	0	1	0	1	0	1	0

Figure 2

Instead of referring to Figure 1, we can use the square in Figure 2 to describe our seven-point geometry. When a point is on a line, we enter a "1" in the column labeled by this point and in the row labeled by this line; if the point is not on the line, we enter a "0" in this row and column. A square array of numbers like this is usually called a *matrix*. The matrix in Figure 2 is called an *incidence matrix* for the given seven-point geometry because it shows which points and lines of this geometry are "incident," that is, which points lie on which lines and which lines contain which points.

We can notice several things about this seven-point geometry. First, we have already seen that it has exactly seven lines. Second, each line has the same number of points on it, namely three; and each point is on the same number of lines, again three. These observations are all part of the following theorem, which would not be hard to prove if we had more time. (You can probably prove the first statement for yourself, and maybe the second, too!)

THEOREM: *For every finite projective plane* P, *there is a positive integer* n, *called the order of* P, *such that each line of* P *contains exactly* $n+1$ *points, and each point of* P *is contained in exactly* $n+1$ *lines. Furthermore,* P *has exactly* n^2+n+1 *points and exactly* n^2+n+1 *lines altogether.*

Our seven-point projective plane is of order 2: there are $2+1=3$ points on each line; there are $2+1=3$ lines containing each point; and there are $2^2+2+1=7$ points and $2^2+2+1=7$ lines altogether.

Does there exist a finite projective plane of any given order n? The complete answer to this question is as yet unknown. In 1906, Veblen and Bussey showed that *there exists a projective plane of order* n *whenever* n *is a power of a prime number.* As of now, projective planes of no other orders are known, though, so far, attempts to prove these the only possible orders have failed. Thus, the values of n below 20 for which projective planes of order n are known to exist are the following:

$$2=2^1$$
$$3=3^1$$
$$4=2^2$$
$$5=5^1$$
$$7=7^1$$
$$8=2^3$$
$$9=3^2$$
$$11=11^1$$
$$13=13^1$$
$$16=2^4$$
$$17=17^1$$
$$19=19^1.$$

There are infinitely many powers of primes, so there are infinitely many positive integers n which are orders of projective planes. It is now known that there are also infinitely many positive integers n which are *not* orders of projective planes. The first, and for a long time the only, such n known was $n=6$. That there is no projective plane of order 6 turned out to follow from a discovery, made in 1901 by the French mathematician M. G. Tarry, about the apparently unrelated subject of Latin squares.

LATIN SQUARES

Now we turn our attention to a seemingly unrelated topic, Latin squares. The easiest way to grasp the concept of a Latin square is to look at some examples. Here is a Latin square of order 3:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1. \end{array}$$

By definition, a Latin square of order 3 has three rows and three columns, and each row and each column contains each of the same three symbols exactly once. In this example, the three symbols are the numerals 1, 2, and 3, but any three

distinguishable symbols would do just as well. Historically, the name "Latin square" derives from the early practice of using Latin (as opposed to Greek) letters as symbols in writing such squares. Here are two Latin squares of order 4:

1	2	3	4		1	2	3	4
2	1	4	3		3	4	1	2
3	4	1	2		4	3	2	1
4	3	2	1		2	1	4	3.

Each of these squares has four rows and four columns, and each row and each column contains each of the numbers 1, 2, 3, 4 exactly once. Latin squares of order 5 or 6 or any higher order are defined in a similar fashion.

The two Latin squares of order 4 shown above are related to each other in a special way. This is easier to see if we form a combined square, as below, whose elements are the elements of the two given squares written in pairs:

1, 1	2, 2	3, 3	4, 4
2, 3	1, 4	4, 1	3, 2
3, 4	4, 3	1, 2	2, 1
4, 2	3, 1	2, 4	1, 3.

If we agree to count the pair 3,2 as being different from the pair 2,3, etc., we can observe that in this combined square *no pair appears more than once.* Now there are just $4 \times 4 = 16$ possible pairs altogether (because there are the four possibilities 1,2,3,4 for the first number of the pair and the same four possibilities for the second). Hence, since no pair appears more than once, *each possible pair appears exactly once in the combined square.* When this happens, the two Latin squares are said to be *orthogonal.*

Orthogonal Latin squares are useful in the efficient planning or designing of experiments. Suppose, for example, that in a factory we wish to test four different machine operators with four different machines, pairing operators and machines in all possible ways. We could use our orthogonal Latin squares of order 4 to do this. On the first day of the test we could pair men and machines according to the top row

1, 1 2, 2 3, 3 4, 4

of our combined square, that is, man 1 with machine 1, man 2 with machine 2, etc. On the next day we could pair them according to the second row

2, 3 1, 4 4, 1 3, 2,

that is, man 2 with machine 3, man 1 with machine 4, etc. In this way, each man would be paired with each machine exactly once during the experiment, without ever leaving a man or a machine idle. In fact the orthogonal Latin square design accomplishes a bit more. The four columns in the combined square may be thought of as designating four rooms, each to be occupied by one man-machine pair. This results in assigning men and machines to rooms in a "balanced" manner in that each man, and each machine, is assigned to each room exactly once during the experiment.

Orthogonal Latin squares have been known for a long time. They were first studied by the eighteenth-century Swiss mathematician Leonhard Euler. In the year 1779, he wrote a paper on Latin squares in which he posed the problem of finding two orthogonal Latin squares of order 6. He phrased this problem in the following interesting way: Can you arrange 36 army officers, one from each of 6 ranks in each of 6 regiments, in a square on a parade ground in such a way that in each row of the square, and in each column of the square, there appears a man of each rank and a man of each regiment? This problem came to be called the "problem of the 36 officers." For over a hundred years people tried to solve it. In the year 1901, Tarry showed that it was impossible to solve it, that is, Tarry proved that there are *no* orthogonal Latin squares of order 6. His proof wasn't very elegant; it consisted of checking hundreds of cases and took many hours to verify carefully.

Still, it saved people many more hours that they might have spent in trying to solve an unsolvable problem!

When Euler posed the problem of the 36 officers in 1779, he had already tried to construct orthogonal Latin squares of many different orders. He was able to do this for all the odd orders 3, 5, 7, 9, 11, and so on (You may be able to, too!), and also for all even orders that are multiples of 4, like 4, 8, 12, 16, 20, and so on. He was, however, unable to do this for any of the intervening even orders, like 2, 6, 10, 14, 18, etc.; and he therefore conjectured that no orthogonal Latin squares of these orders exist. Subsequently, many mathematicians tried to prove or disprove Euler's conjecture. It was still unresolved in the first decade of this century, when a German mathematician, P. Wernicke, worked out what he thought was a proof of Euler's conjecture. His proof was published in 1910 in a respected German mathematics journal, and this seemed to settle the matter. In 1921, however, an American mathematician, H. F. MacNeish, published his finding that there was an error in Wernicke's proof; and in the next year he offered a proof of his own along quite different lines. This turned out to be wrong too! You can see that Euler's conjecture presented a very tricky problem.

Euler's conjecture is obvious for order 2, as you can easily prove for yourself; and, 122 years after Euler made his conjecture, Tarry laboriously proved it for order 6. This left order 10 as the first unsettled case: do there, or do there not, exist orthogonal Latin squares of order 10? At the University of California at Los Angeles in the summer of 1952, a group of mathematicians worked on this problem, among others. One of these mathematicians, James Ward, used a modification of Tarry's method of checking cases. To have done this by hand would have been hopelessly laborious, but Ward had at his disposal what was then one of the world's fastest electronic computing machines, the SWAC. After over one hundred hours of machine search, millions of cases had been tested; but these were only a tiny fraction of the possible cases, and no orthogonal Latin squares of order 10 were found. Had they been, this would have constituted a striking instance of the use of an electronic computer in proving a new mathematical theorem.

Euler's conjecture was finally completely settled in the spring of 1959 by E. T. Parker, R. C. Bose, and S. S. Shrikhande. You may remember reading about this discovery, which made the front pages of newspapers at the time. It was also written up in a highly readable article in the *Scientific American* for November, 1959, which I recommend to you. The result was very surprising. Euler, great mathematician though he was, was *wrong in all the previously unsettled cases:* the fact is that orthogonal Latin squares of all the orders 10, 14, 18, 22, and so on, *do exist.* Only for orders 2 and 6 do they fail to exist!

Conclusion

Now what do these discoveries about Latin squares have to do with finite projective planes? Before we answer this question, let us mention one more fact about orthogonal Latin squares. Sometimes it is possible to find several Latin squares, all of the same order, and each of them orthogonal to each of the others. It is not very hard to see, however, that for any given order n there can never be more than $n-1$ such mutually orthogonal Latin squares. (Try to prove this for yourself!)

The connection between projective planes and Latin squares is given by the following theorem:

Theorem: *There exists a finite projective plane of order* n *if and only if there exist* $n-1$ *mutually orthogonal Latin squares of order* n.

This theorem takes a bit of proving, and about all we can say about the

proof here is the following. Recall that a finite projective plane of order n can be described by its incidence matrix, which has n^2+n+1 rows and columns. It turns out that this incidence matrix can always be constructed by using $n-1$ orthogonal Latin squares of order n in a certain fashion. Conversely, it also turns out that $n-1$ Latin squares of order n can always be extracted from the incidence matrix of a projective plane of order n.

Now Tarry's result assures us that there cannot be two orthogonal Latin squares of order 6, and, hence, there certainly can't be five mutually orthogonal Latin squares of order 6. Therefore, by the above theorem, there is no finite projective plane of order 6. In 1949, two mathematicians, R. H. Bruck and H. J. Ryser, were able to find *infinitely many* values of n for which there is no projective plane of order n.

There are, however, still infinitely many values of n left undecided. In particular, it is still an open question whether or not there is a finite projective plane of order 10. By the above theorem, there is such a plane if and only if there are nine mutually orthogonal Latin squares of order 10. To date, only two have been found, and no one knows whether or not there are more. Maybe someone will find out tomorrow!

What's new?

BOOKS

College

Advanced Euclidean Geometry (Modern Geometry), An Elementary Treatise on the Geometry of the Triangle and the Circle, Roger A. Johnson. New York: Dover Publications, Inc. Paper, ix+318 pp., $1.65.

An Elementary Treatise on Curve Tracing, Percival Frost. New York: Chelsea Publishing Company, 1960. Cloth, xvi+210 pp., $3.50.

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First Course in Algebra, Arthur W. Weeks and Jackson B. Adkins. Boston: Ginn and Company, 1961. Cloth, ix+534 pp., $4.40.

Mathematics in Daily Use (3rd ed. rev.), Walter W. Hart, Veryl Schult, and Lee Irvin. Boston: D. C. Heath & Company, 1961. Cloth, ix+374 pp., $4.20.

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The Process of Education, Jerome S. Bruner. Cambridge, Mass.: Harvard University Press, 1960. Cloth, xvi+97 pp., $2.75.

Square circles

FRANCIS SCHEID, *Boston University, Boston, Massachusetts.*

Square circles can be constructed provided the foundation is suitable.

NOWADAYS all mathematical literates know that Euclid was wrong. At least, like any other mortal, he was not entirely right. His method was inspired, his execution brilliant, his intuition powerful, but his logic was not entirely flawless. Speaking from the comfortable advantage of two thousand years of hindsight, we can say that Euclid used results which he had neither formally assumed nor formally proved. These results seemed so obvious (For example, an arc joining a point inside a given circle to a point outside the circle must somewhere have a point in common with the circle.) that they were used where needed without either thought or mention. And in such oversights Euclid was, of course, wrong.

There is a certain amount of satisfaction in criticizing the giants of the past. It brings a pleasant, temporary feeling of superiority. Astronauts of the twenty-first century may view with slightly upturned noses the impressive efforts of today's struggling pioneers. But criticism, like stones, should be cast only by those who are themselves faultless and whose houses are of sturdier stuff than glass.

Disparaging remarks about square circles in the recent mathematical literature lead one to wonder if the lesson of Euclid is still forgotten upon occasion, even by moderns who have learned it well. An excellent recent volume on algebra asserts that the very thought of square circles does violence to both logic and common sense. And a fine effort to provide modern materials for the growing numbers of embryo mathematicians in our secondary schools singles out the square circle as an example of what is clearly impossible. In both these cases, the nonexistence of such circles seems to be accepted, not on the basis of assumptions listed or results proved, but as something it would be ridiculous to question. If so, this would be Euclid's error all over again.

Two examples should now suffice to initiate a chain reaction of creativity and produce a heavy fall-out of square circles. In the spirit of first things first, however, we must begin by stipulating that "A *circle* is the set of points, in a plane, equidistant from a given point," or words to that effect. Here the concepts of point, plane, and distance are abstractions. In these relativistic times, the number of humans who insist that only the original Euclidean interpretation of these terms is proper and sane dwindles annually. Those who remain dogmatic about it, when challenged to explain what, for example, a point actually *is*, bring forth a confusion of words but nothing of any more substance than Euclid's desperate "A point is that which has no parts." (This bit of escapism may look appealing, but anyone on the verge of deciding that it is for him should be told that Einstein and most of the world's mental heavyweights will be on the opposite team.) Converts to the theses that the faster we move, the more slowly we age and the heavier we get, that it is senseless to speak of simultaneous events, that our bodies are not smooth connected things but clouds of high-speed electrical particles totally separated from one another, and that maybe somewhere creatures really do look out upon the world through X-ray eyes, have

learned to re-examine the image of the universe which they have personally been constructing since childhood. For such moderns the following examples will be child's play:

Taxicab geometry. A taxicab travels on a network of roads, a typical part of which is shown in Figure 1. Using mathematical license (and following Euclid), we imagine these roads and taxicabs to have no thickness and to consist of Euclidean points. This network will be our plane. The existence of other "points" not on the roads is not recognized, not by taxicabs anyway. It is the lines of Figure 1, not the spaces, which concern us. As for *distance*, it is only common sense to use this word for a quantity measured *along the roads* as the taxicab goes, not as the crow flies. In skyscraper country even the crow may find our concept of distance quite useful. Thus, points A and B are at a distance of three blocks from one another, to use the most convenient unit, whereas the distance from B to C is seven blocks.

Now, where are all the points at a distance of three blocks from A? It is a simple matter to locate them. Figure 2 gives the picture. Behold the square circle!

But not so fast. We must remember the lesson of Euclid and avoid too hasty conclusions. The locus of Figure 2 is unquestionably a *circle*, simply by definition. Its center is A, and its radius three. But is it also a square? What is a square anyway? And suddenly it becomes clear that there is an even more basic question. Squares involve straight line segments. So, what is a straight line? But this is easily settled. It would cause too much heartbreak if a straight line segment were to be anything but a path of shortest length between two points. Adopting this familiar definition, we soon realize that between two given points of our network there are quite a number of straight lines. From A to B, for example, there are three, as Figure 3 shows. This issue is worth further attention, and we shall return to it a bit later. At present, the subject of prime interest is squares, and we are in serious trouble. Our entire circle has only twelve points, not enough to even dream about forming straight lines, let alone squares. Plainly a square can be formed with "vertices" at 1, 2, 3, and 4 and it will include all the points of our circle, but it will include other points as well. The points of our circle are *on* a square, but they do not make up the whole square. Our success is incomplete.

The previous paragraph is not, of course, thorough or rigorous geometry. Among other things, the word *path* has not been carefully introduced. Anyone interested can take a few moments to brace the foundations, and may then explore taxicab geometry to his heart's content. Here, however, we shall turn to a second example:

The king's walk. On a chessboard the king walks (majestically) one cell at a time, each step being to an adjacent cell. For example, a king at A in Figure 4 could step to any of the numbered cells, 1 to 8. The chessboard is now taken as our second interpretation of a *plane*. The cells will be our *points*. Note that it is now the spaces, not the lines, on which we must focus. An ordinary 8-by-8 board will do, but those whose imaginations require a larger radius of action may choose a larger board, even an infinite one if they wish. The *distance* between two points (cells) will be the smallest number of steps by which a king may proceed from one point to the other. The distance from A to B in Figure 4 is therefore three. Once again a straight line segment will be a path of shortest length joining two points. Once again, it is plain that between two given points there may be several straight lines. From A to B, for example, there are three.

Now, where are all the points which are at a distance of three from A? It is a simple matter to locate them. Figure 5 gives the picture. Behold the square circle!

This time our success is complete. The locus of Figure 5 is a circle by definition.

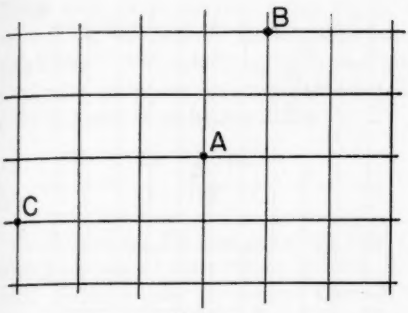

Figure 1

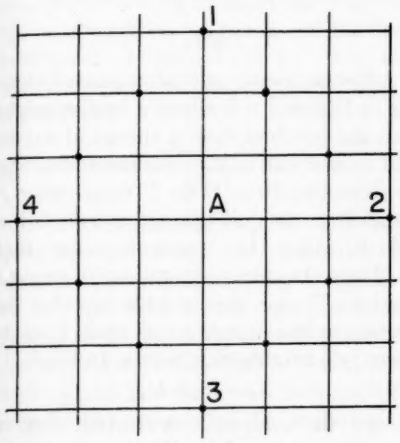

Figure 2

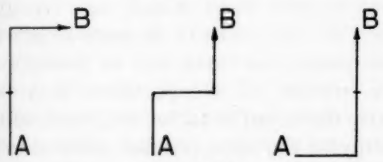

Figure 3

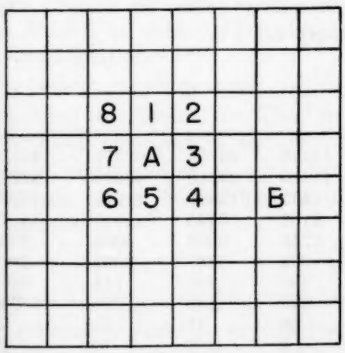

Figure 4

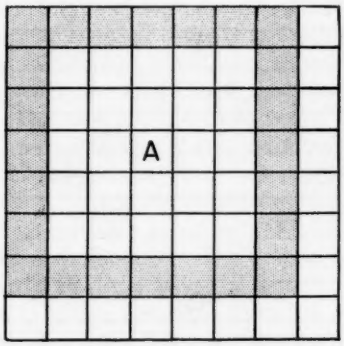

Figure 5

Its center is A, its radius three. There are twenty-four points on its circumference. These twenty-four points also form four straight line segments of length six, meeting at obvious vertices to form a quadrilateral. Since the diagonals of this quadrilateral are of equal length, *again six*, there is sufficient justification for calling it a square. Certainly it looks like a square.

Now that our main purpose is achieved, a moment spent as the ounce of prevention may make the pound of cure unnecessary.

First, the obvious fact, that a line joining points on opposite sides of a given line must have a point in common with the given line, is incorrect. The two long diagonals of the chessboard in Figure 4 constitute a counter example.

Second, the obvious fact that there can be only one straight line joining two given points is obviously incorrect. Both of our examples show this, and we shall count the number of such lines shortly.

Third, the obvious fact that there is no such thing as a biangle, (a polygon with just two distinct sides), is incorrect. This follows at once from the last paragraph, for any two straight lines joining the same pair of points form a biangle. Figure 6 shows a delightful biangle, with vertices at 1 and 2. It takes will power to avoid noticing that this biangle is at the same time a square, with vertices at 1, 2, 3, 4. Moreover, it is also a triangle, with vertices at, say, 1, 2, 3. (The length of

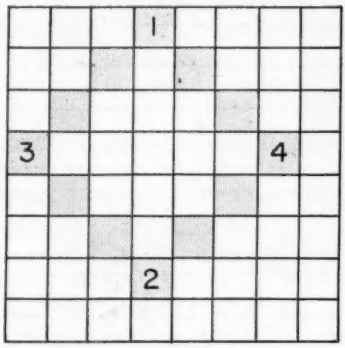

Figure 6

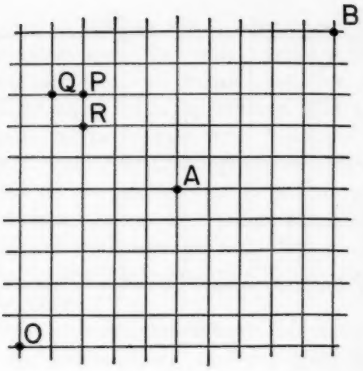

Figure 7

side 2-1 of this triangle is the sum of the lengths of the other two sides.) The chess king's walk has led us not only to a square circle, but also to *a square triangle*, even to a square triangular biangle. It would be pleasant to be able to report that this versatile locus is also circular, but unfortunately this is not the case. Whether triangular circles or biangular circles exist will be left an open question.

For a final effort let us explore a little further one of the issues raised above, the number of straight lines joining two given points. Taking taxicab geometry first, we may choose some point O as reference and ask how many straight lines join it to various other points to be called P. By symmetry it is sufficient to restrict P to the "first quadrant." As a test of your intuition you may try to estimate the answers for some particular points, say the A and B of Figure 7. Point B is just ten blocks north and ten blocks east of O.

A few moments of contemplative viewing of Figure 7 will afford a better insight into the problem than a thousand words, and should lead to the conclusion that any straight line from O to P must enter P either from the left (via Q) or from below (via R). After this, one soon realizes that to obtain the required number (of straight lines to P), one simply adds together the corresponding numbers of such lines to the two neighboring points. In brief

$$N_P = N_Q + N_R. \tag{1}$$

To get the computation started, observe that from O to any point directly to the north or east there is only one straight line. For, on a path of shortest length to such points, no turns can be permitted. The results of computation may be neatly displayed in table form, each entry occupying the same position relative to O as its corresponding point in Figure 7. The boundary values in Table 1 are accordingly all 1's.

TABLE 1

TAXICAB NUMBERS

1	11	66	286	1001	3003	8008	19448	43758	92378	184756
1	10	55	220	715	2002	5005	11440	24310	48620	92378
1	9	45	165	495	1287	3003	6435	12870	24310	43758
1	8	36	120	330	792	1716	3432	6435	11440	19448
1	7	28	84	210	462	924	1716	3003	5005	8008
1	6	21	56	126	252	462	792	1287	2002	3003
1	5	15	35	70	126	210	330	495	715	1001
1	4	10	20	35	56	84	120	165	220	286
1	3	6	10	15	21	28	36	45	55	66
1	2	3	4	5	6	7	8	9	10	11
0	1	1	1	1	1	1	1	1	1	1

Note that each interior entry is, as it should be, the sum of its left and lower neighbors. We conclude, for example, that the numbers of straight lines to points A and B of Figure 7 are

$$A \quad 252$$

$$B \quad 184{,}756.$$

Now plainly, the computation may be continued and a larger table produced. But it is difficult not to notice that it is Pascal's triangle which is being generated, in a somewhat different orientation from the usual. All our numbers are binomial coefficients. Indeed, the entry which is n rows above and m columns to the right of O is actually

$$\frac{(n+m)!}{n!m!} \tag{2}$$

or $C\ (n+m,\ n)$. This number is perhaps more familiar when interpreted as the number of combinations of n things in a set of $n+m$. We would have come to this result directly had we begun by observing that any straight line from O to a point n blocks north and m blocks east must be $n+m$ blocks long, since no westward or southward travel can be permitted, and that two such lines differ only in the driver's choice of when to include the n northward blocks. Putting them all first carries him around one boundary of the rectangle which has O and the target point at opposite corners; alternating them leads him, roughly speaking, through the middle. It is a question of which of the $n+m$ blocks he chooses for the northward combination.

The same numbers have, of course, many other interpretations. They appear in the most unsuspected corners of the world of mathematics. A very familiar appearance is as the coefficients in the expansion of

$$(1+t)^{n+m} \tag{3}$$

by powers of t. The expression (3) is known as a *generating function* of $C(n+m,\ n)$, which appears as the coefficient of t^n.

Which type of formula, (1), (2), or (3), is most efficient for actually producing the numbers $C(n+m,\ n)$? Rather clearly it is (1), which we used. Each new entry in our table cost us just exactly one addition, an amazingly low price. Computation of this sort, in which new entries are produced by a fairly simple process from entries previously computed, is known as *recursive* computation. The process itself, in our case (1), is called a *recursion formula*. Plainly, recursion formulas make for relaxed and efficient computing.

And now finally, let us consider the same enumeration problem for the king's walk. Again we choose some point (cell) as reference O and ask how many straight lines join it to various other points to be called P. By symmetry, P may be restricted to the quadrant shown in Figure 8. Once again you may wish to give your intuition a workout. How many straight lines are there from O to A? To B?

Another moment for meditation has arrived. Figure 8 and the honest effort which traditionally brings its own reward will eventually carry the (correct) conviction that any straight line from O to P must enter P from one of its three immediate lower neighbors, Q, R, and S. This implies that to obtain the number of straight lines to P one simply adds together the corresponding numbers for these three neighboring points. In brief,

$$N_P = N_Q + N_R + N_S. \tag{4}$$

This *recursion formula* will produce results beautifully, but it needs starting values and boundary values to feed upon.

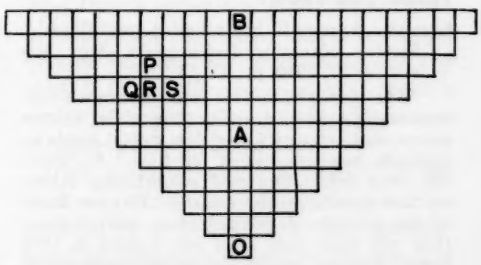

Figure 8

TABLE 2

KING'S WALK NUMBERS

1	10	55	210	615	1452	2850	4740	6805	8430	9073	8430	6805	4740	2850	1452	615	210	55	10	1
	1	9	45	156	414	882	1554	2304	2947	3179	2947	2304	1554	882	414	156	45	9	1	
		1	8	36	112	266	504	784	1016	1147	1016	784	504	266	112	36	8	1		
			1	7	28	77	161	266	357	393	357	266	161	77	28	7	1			
				1	6	21	50	90	126	141	126	90	50	21	6	1				
					1	5	15	30	45	51	45	30	15	5	1					
						1	4	10	16	19	16	10	4	1						
							1	3	6	7	6	3	1							
								1	2	3	2	1								
									1	1	1									
										0										

Observe, however, that from O to any point on either diagonal there is only one straight line. A path of shortest length to such points cannot leave the diagonal. The boundary values are all 1's. Table 2 now gives some results, each entry corresponding to the similarly located cell of Figure 8. Note that each entry is the sum of its three lower neighboring values.

Although the recursion formula (4) is perfectly satisfactory for producing the king's walk numbers, a natural human instinct seems to call for an expression similar to (2), which is not as useful a product as its competitor (1) but appears better packaged. One possible route to such a formula is the observation that

$$(5) \qquad (1+t+t^2)^m$$

is a *generating function* for king's walk numbers. When (5) is expanded, the coefficients are those entries of Table 2 in the mth row above O. A little forceful algebra is now in order to exploit this fact and produce a general formula for these coefficients, but this will be left to those desperately anxious to see the result.

What is to be concluded? It is certainly not that square circles are vital elements of modern technology, though the history of our subject reminds us often enough that it is perilous to predict the path of applied mathematics too far over current horizons.

Our conclusion is not new, but it will bear infinite repetition. In any branch of abstract mathematics we accept only those facts which follow from the assumptions made. And assumptions made must be openly declared, insofar as this is humanly possible. We do not keep secrets from ourselves or from one another. (This was Euclid's error.) The carpenters who build the superstructure must be told the precise extent of the foundations. Square circles can be constructed, provided the foundation is suitable. They are not illogical, or contrary to common sense, or ridiculous. They are merely *non-Euclidean*, and the atmosphere of heresy which formerly followed this term has long since vanished, the result of two centuries of intellectual scrubbing (brain-washing, if you prefer).

Have you read?

SHEPARD, JOSEPH K. "Our Split-Second World," *Sunday Magazine Indianapolis Star*, November 27, 1960, pp. 11–16.

"The march of science into tomorrow's cosmos needs ever-finer calibration of its instruments, and among its precision tools it needs an accurate measuring stick for time." So starts this very interesting and enlightening article on time and its precise measure. Did you know of the recently developed atom cesium clock that will only vary .5 of one second in 1000 years? And yet this is not precise enough, and science hopes to reduce the error to .5 of one second in 300,000 years. It may come as no surprise to you that 90 per cent of all clocks are now electric, that your timekeeper is the power plant, that alternating current is standardized at 60 cycles per second and varies less than .02 of one cycle in 24 hours, and that radio station WWV broadcasts only time tones from which all dynamos are paced.

Shake your arm to keep your old-fashioned, inaccurate, self-winding watch running and read the article. Your students will like it also.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

If not solid geometry, then what?

SEYMOUR SCHUSTER, *Carleton College, Northfield, Minnesota.*

Is the answer a study of vectors and a vector approach to geometry?

FOR SEVERAL YEARS now we have heard arguments in opposition to the standard one-semester course in solid geometry. We've heard that the techniques and arguments of solid geometry are merely slight extensions of those the student learns in plane geometry. We've heard that the subject matter is relatively useless, and that other choices of subject matter would be more pertinent. In short, a semester of the twelfth year devoted to solid geometry is not a wise utilization of the time.

Several alternatives have been mentioned, supported, and even tried in many quarters. Among these are statistics and/or probability, calculus, more school algebra, groups and fields, matrices and linear algebra.

The purpose of this note is to provide still another alternative as a replacement for the old stand-by of synthetic solid geometry. Our choice is not the result of scanning the market shelf for a new model to replace our "outmoded" one. We prefer to present it as a choice which satisfies three basic conditions. An analysis of the complaints (mentioned in our opening paragraph) against solid geometry provides motivation for the first two conditions, and further discussion will give us the third.

Condition 1. *The course should cover material that is useful relative to the general stream of a student's mathematical training.*

Condition 2. *The subject matter should acquaint the student with mathematical techniques and arguments that are new to him.*

For the past quarter-century or more, the trend, in the United States at least, has been to give geometry a low-priority status in the colleges and universities. While geometry had a nice garden of theorems to be picked, topology and algebra provided the research mathematicians and their Ph.D. candidates with virtual plantations. Who wants a backyard garden when a plantation is available? Moreover, the emphasis—perhaps overemphasis—on generalization served to further establish geometry in its low-priority niche. Surely, it is true that geometry has many general results, but these had not been its feature attractions; here also, it could not compete with algebra, modern analysis, or topology. The result was the gradual disappearance of advanced geometry courses from the collegiate curriculum.

Another aspect of this trend has been the removal of analytic geometry as a course offering. This is an obvious consequence of streamlining the curriculum through integrating the subject matter of calculus with analytic geometry. While I am a strong proponent of such streamlining and the unifying of course content, I confess that it quite often happens that our students do not become as intimate with the analytic geometry as they should. And this is sometimes due to negligence in our teaching rather than to inadequacies on the part of students. In any case, it is not the exception that there is a shortchanging through the elimination of some analytic geometry.

This trend—putting geometry in the back seat—is quite unfortunate, for it has resulted in a deficiency in the development of geometric insight in too many of our mathematics students. This, in turn,

is felt strongly in all quarters where students of science and engineering are being trained. What has been observed, especially, is the inability of students to think clearly about nonplanar problems. One of the world's leading figures in X-ray crystallography, who has taught in Germany, Ireland and the United States, constantly deplores the fact that his American students, many of them quite gifted, never come to him knowing any space geometry. Yet, we are eliminating solid geometry, at present the only course in which a high school student gets any formal training in thinking about problems of space. This unsatisfactory state of affairs leads us to the third stipulation for the course we suggest as a possible replacement for solid geometry.

Condition 3. *The course content should be principally geometry, including some three-dimensional geometry.*

The proposal

The course we suggest is a *vector approach to geometry.* A possible syllabus follows in a later paragraph.

Let's now reflect on whether this proposal does actually satisfy the three conditions we laid down. As for Condition 1, no one would deny the useful and fundamental role of vectors in the various branches of mathematics. A knowledge of vectors is quite necessary for physicists, engineers and social scientists who wish to utilize mathematics as a tool for analysis; and pure mathematicians, whether they are geometers, analysts, algebraists or topologists, find vectors essential.

Condition 2 will be satisfied for many reasons. A new concept is introduced, and much of the student's mathematical world is to be viewed through this new medium. Geometric understanding must be translated into vector (and algebraic) language, and translations in the reverse directions must also be carried out. The vector approach can be exciting, without being overwhelming, if it is introduced as a follow-up of tenth-year geometry. *It should therefore be introduced as a new way of looking at old things.* And, after some of the basic techniques are understood and exploited to prove, in an easy manner, results that were difficult to obtain in the "old-fashioned" way, then the path is open to applying the techniques to new matters with geometric content. The new geometric material could well be co-ordinate geometry via vectors.

The beginning of the course could be devoted to co-ordinate-free vectors. This has mathematical as well as pedagogical advantages. Co-ordinate systems can then be introduced as another instrument for solving geometric problems. They are not inherent but are rather imposed on a problem in such a manner as to facilitate its solution. The advantages of rectangular co-ordinate systems will be obvious to the student who learns about dot and cross products. (Observe that we are already dealing with three-dimensional geometry, thus satisfying the demands of Condition 3.) Without any further tools of vectors, the analytic geometry of the line, plane, circle, and sphere could be treated.

Several interesting mathematical paths which can follow the introductory material suggest themselves, but we postpone discussion of these until after presenting a syllabus for the basic course.

A possible syllabus and extensions

1. Definition of vector as a directed line segment.
2. Addition of vectors.
3. Multiplication of a vector by scalar (real number).
4. Subtraction of vectors.
5. Linear combinations of vectors.
6. Division of a segment in a given ratio.
7. Right-handed triples.
8. Rectangular co-ordinate systems and bases.
9. Co-ordinate solutions (utilizing vectors) to problems done earlier in co-ordinate-free manner, e.g., finding the median point of a triangle.

10. Applications to the complex plane.
11. Dot products.
12. Projections and components.
13. Physical applications.
14. Analytic geometry of the line in the plane.
15. Parametric representations of the line.
16. Parallelism and perpendicularity.
17. Distance.
18. Distance from a point to a line.
19. Analytic method of proof.
20. Circles.
21. Spheres.
22. Direction cosines and direction numbers.
23. Analytic geometry of planes.
24. The line in three dimensions.
25. Angle between two lines.
26. Angle between two planes.
27. Cross products.
28. Area.
29. Distance between two skew lines.

Depending on the length of time available and the ability and interest of the students, the instructor may choose to go in any of the following directions:

1. Trigonometry via vectors. (Spherical, as well as plane, is readily handled by vectors.)
2. Higher geometry in the form, for example, of the theorems of Desargues, Pappus, Menelaus, and Ceva.
3. Plane curves; e.g., conics, involute of circle, cycloid and trochoid.
4. Graphing linear inequalities and elements of linear programming. (Interesting applications to social science are possible here.)
5. Convexity.
6. Rigid Motions.
7. Vector Spaces.
8. Matrices.

These few suggestions all admit to relatively easy handling of vectors.

Incidental virtues

While the standard solid geometry course was nearly devoid of algebra, and while current twelfth-year courses in algebra have virtually no geometry, the vector course we propose is strong in both algebra and geometry. Nearly all, if not actually all, the mathematics that was learned by the high school senior, prior to our proposed course, must be brought to bear on it. Thus, the student gets a *review through utilization.*

In addition, there are golden opportunities to treat a number of other topics in an incidental and completely natural manner. A few examples are:

1. *Inequalities* and their geometric interpretation.
2. *Sets* as loci, defined by equations and/or inequalities, leading to applications of set union and intersection.
3. *Statics,* applying components and projections.

If the class is somewhat gifted, the instructor might take a crack at something more sophisticated by making the distinction between *affine* geometry and *Euclidean* geometry. Affine theorems and affine properties may be viewed as those obtainable without the introduction of a dot product, while the geometry that relies on the dot product (angle measure, perpendicularity, etc.) is Euclidean. By means of the very definition of vector equality, it can be seen that lengths can be compared in affine geometry only if the lengths are measured on a single line or on parallel lines. For comparing two lengths on nonparallel lines, one must cross the boundary and move into Euclidean geometry.

Finally, we would add that sermons on the power of abstraction are very much in order all along the way, especially if the instructor wishes to bring in abstract algebraic concepts. For example, if *vector space* becomes the focus, then nongeometric examples such as polynomials and fields with elements of the form $a+b\sqrt{k}$ (with a and b rational, and k square-free) may be chosen. With examples of this sort, the student can gain new insights while reviewing.

The place of probability and statistics in the mathematics curriculum of the senior high school and junior college

H. E. WILLIAMS, *Vanderbilt University, Nashville, Tennessee.*

Should probability and statistics be offered at the secondary-school level? If so, how much emphasis should be given?

IN RECENT YEARS there have been considerable discussions and experimentation concerning the teaching of probability and statistics in the senior high school and junior college. The majority of the discussion is pitched toward the twelfth grade and the freshman or sophomore college years where an overlapping of ideas frequently results; for this reason, it was decided to include both of these levels in the scope of this article. It is the intention of the article to present and analyze the factors that have brought about this flurry of discussion, to present the situation as it stands at present with regard to this subject, and to survey the transition that has already commenced, along with the outlook for the future.

THE PRESENT STATUS OF PROBABILITY AND STATISTICS

We are living in a society which is most assuredly being influenced continually by statistics. Descriptive statistics confront the public daily in the form of polls, surveys, and cross-sectional samplings appearing in the newspapers and magazines. Graphic portrayal of economic and sociological statistics is commonplace. On a more sophisticated plane, educators, social scientists, physicists, biologists, and industry have found widespread uses and applications for statistics as the mathematicians have explored and exploited many new facets of this field. Statistical inference has achieved newly found acclaim as our best strategy for exploring the ramifications of massed data.

At one time, the word "statistics" signified numerical observations concerning the State. Statistics in this context can be traced back more than four thousand years to the Chinese. Today, that definition has been made more general and extended to include the study of phenomena being produced in large numbers. The statistical method consists of the enumeration of the elements of a certain set into first their classification by certain attributes, then into the establishment of a function representing their distribution, and finally, in the determination of relations between phenomena.

Interest in the theory of probability is generally conceded to have started with the French. Games of chance being popular there in the seventeenth century, there was a considerable interest in the computation of odds. In the course of this interest in odds, challenging problems arose which led to an exchange of correspondence between the eminent mathematicians, Fermat and Pascal, from which grew much of probability theory.

The wedding of probability and statistics might well be traced to Jacques Bernoulli. His "Art of Conjecturing," published in 1713, served as the cornerstone about which the structure of mathematical statistics has developed. Quételet, in

the early nineteenth century, is credited with first having applied Bernoulli's work to research on the social laws of mankind.

From these beginnings, probability and statistics have grown to the stature of a basic tool in the study of the social and natural sciences and the problems of government and industry. Genetic studies, studies of various types of randomly occurring events, and particle behavior, are areas currently being probed by the use of probability and statistics and are creating a demand for trained personnel.

A highly significant modern application of probability and statistics is found in industry. Statistical quality control is concerned with the problems of: (1) analyzing a product specification on a small sample of the product and mathematically inferring the quality of the specification for an entire lot (often quite large) from which this sample was drawn. This technique is extremely useful in the interchange of products from a producer to a consumer and has come into widespread use since its introduction during World War II, at a time when manpower was critical; (2) monitoring the quality of a production process, by again analyzing a specification from a small sample of the product and inferring mathematically the quality of the specification on the entire production process for a given period of time. This technique has been of great value to industry in saving time and manpower. This field of application of the theories of probability and statistics is still relatively in its infancy and expanding daily by leaps and bounds, calling for more persons trained in probability and statistics, both on the management and labor levels. It is not uncommon to find persons from the labor force enrolled in night-school courses in statistics to help provide a background for this type of work. The emergence of statistical quality control methods has created a demand for probability and statistics in the curriculum of engineering schools, where little demand existed ten years ago.

For some time now, the economists have been making use of the ideas of statistics and statistical inference in market fluctuation studies and time series studies. Need for trained persons grows steadily.

Another area where vast usage of statistical inference has developed in the last twenty-five years is in the analysis of experimental results by persons engaged in psychological, sociological, biological, and medical research. Graduate students in these fields are often required to attain competence in statistical usage.

With the early reverberations of the above considerations being felt, the National Council of Teachers of Mathematics was prompted as far back as 1940 to state, "From its very modest estate of a few years ago, the theory of statistics has grown significantly and has been successfully employed, for instance, in education and sociology. Statistical concepts can now be regarded as a necessary qualification for many activities." [10] This statement could be considerably magnified to give an accurate picture of the situation twenty years later, so rapid have been the advances during the intervening years.

Let us now consider how probability and statistics are presently incorporated into the educational curriculum of secondary schools. Probability theory has long been taught in college algebra courses but seldom in high school courses. However, some of the more recent texts in college algebra have dropped this topic in favor of others. On the other hand, a trend has arisen recently to include probability and some statistics in college general mathematics and twelfth-grade mathematics courses—a trend that is currently spreading rapidly. S. S. Wilkes, in 1958, gave a very good definitive picture of the situation in regard to statistics in the secondary curriculum: "At the high school level, education in topics in this area is virtually nonexistent, although the new College Entrance Examination Board on Mathematics is now taking the initiative in getting some of it started. At the college

and university level, knowledge in this area has been transmitted mainly outside of mathematics departments through introductory statistics courses at the upper-class and graduate levels in economics, sociology, psychology, biology, and other departments or schools. The method of teaching even the most basic concepts at both levels in many of these departments is largely intuitive and nonmathematical. The ideas are almost completely couched in the language of illustrative examples peculiar to that particular department. The result is that the student does not learn these fundamental concepts in a precise manner and with an appreciation of their full generality.

Let me say, however, that these statistics courses are steadily improving because of the better statistical training being received by the younger men in those departments who are now teaching these courses. But I do not believe that the problem of teaching the basic ideas of statistical inference will ever be solved satisfactorily by introductory statistics courses taught in this intuitive manner at the upper-class or graduate level in departments of a college or university scattered all over the campus." [13]

What is ahead for this subject?

With the ideas of the preceding section in mind, we are led rather naturally to consider the foreseeable future for probability and statistics in the secondary curriculum. In discussing this aspect of the problem, we shall, of necessity, have to incorporate two facets: First of all, we have proposals and recommendations on what should be done in this area made by committees working on the problems of the mathematics curriculum for the secondary schools in the second half of the twentieth century; and secondly, we consider some positive steps that have been made by authors and teachers in this area.

The Committee of the Undergraduate Program of the Mathematical Association of America has been concerned with the problem of probability and statistics during the course of their work. They have recommended a universal course in mathematics for college freshmen and have written some sample materials for this purpose. Their work has been a subject of extreme interest among mathematicians for the past several years. In the second semester of this universal course they recommended the introduction of set theory to be carried over into the development of probability theory and subsequently into a juncture with some concepts of statistics. [5]

The Commission on Mathematics of the College Entrance Examination Board, previously referred to in the quotation from S. S. Wilkes, recommended probability and statistical inference as a good one-semester course for college-capable high school seniors. They have prepared an experimental text for a twelfth-grade course of this type, which has been used with success since 1957. [4]

In addition to these two committees, the Secondary School Curriculum Committee of the National Council of Teachers of Mathematics, in their 1959 report, outlined a college preparatory program in mathematics for grades nine through twelve. The program allowed for flexibility in grade twelve because of the absorption of solid geometry into other parts of the program, and a semester course in probability and statistics was one of three semester courses recommended. [12]

Some good textbook material is available at the present time for courses in general college mathematics that includes topics in probability and statistics, and more texts of this type are appearing every year. Some of these texts can be (and have been) satisfactorily utilized with advanced twelfth-grade classes. The majority of this text material has been prepared within the last five years and has been quite favorably received. This indicates the spreading interest in the introduction of probability and statistics on the secondary level that has been gen-

erated by authors and committees working in this area. Some of the better-known texts are listed among the references.

Two experiments with twelfth-grade students seem worthy of mention. At Madison, Wisconsin, Professor Joe Kennedy experimented with the use of "Principles of Mathematics" by Allendoerfer and Oakley [1] for a course in the twelfth grade at Wisconsin High School. In discussing the course he said, "The last chapter in the book, 'Statistics and Probability,' was very enjoyable. Descriptive statistics, with its simple arithmetical calculations and readily grasped concepts, was quite different from limits and calculus." [8] An experimental unit in probability and statistics was prepared by a teacher, Clarence E. Olander, and taught to an ability-grouped mathematics class. Olander was quite encouraged by the results of the experiment. He stated that many of the concepts are appropriate for all high school students and that, furthermore, most citizens can use an appreciation of the statistical approach because of the need to digest information of a quantitative nature. Secondary students, he feels, need to be aware of the increasingly important role that statistical reasoning is playing in various fields. He goes on to point out that the teacher has a unique opportunity to compare various kinds of reasoning—specifically, to contrast the statistical thinking which is primarily inductive reasoning—with the postulational thinking of formal mathematics. [11]

The question of when in the secondary-school program to introduce probability and statistics is somewhat debatable. Most of the thinking seems to favor either the twelfth grade or the freshman year in college. The problem of which of these two choices is better should be easily resolved. In view of the present immense demand for more persons trained in probability and statistics, the ideal situation would be for material in this area to be offered at both levels. From a practical standpoint, it will probably be some time before this comes to pass, but, as more text materials and more adequately trained teachers become available for use in courses in probability and statistics, we may see this become a reality. As the number of students coming to college with a background in this subject increases, the quality of offerings at the freshman level can then be upgraded.

I think that, from the considerations presented in this article, it is reasonable to assume that in the next few years we shall see more and more probability and statistics appearing in the curriculum of senior high schools and junior colleges, as well as in other places. Also, it is reasonably certain that the demand will continue to increase in the future as more and more applications of statistics and probability, such as game theory and operations research as well as statistical inference, experimental design, and factor analysis techniques, are applied to a wider variety of uses in our lives. This, of course, presents quite a challenge to educators—in particular to departments of mathematics—to develop a curriculum to satisfy these demands. By again referring to S. S. Wilkes, we can arrive at a good picture of the situation facing mathematics departments in the colleges: "The only department in the colleges and universities now sufficiently institutionalized to teach these basic concepts as broadly as they deserve to be taught is the mathematics department. While statistics departments have recently been organized in some of the large universities, there are only a few of them and they are operating at present mainly at the upper-class and graduate levels. However, there are hundreds of colleges and universities which do not have statistics departments, and most of the smaller ones may never have them. In the meantime, the best hope for getting these basic ideas taught effectively and to a sufficiently broad range of students in all of these colleges and universities is through their main sequence of freshman-sophomore courses. Even

though relatively few mathematicians are now familiar with the basic concepts of statistical inference, I am confident that they will find them interesting and that they can pick them up quickly by teaching the material or by attending a summer institute such as those sponsored by the National Science Foundation." [13] I think it goes without saying that in the high schools the mathematics department is the only department prepared at all to offer work in probability and statistics.

The effect of this trend on the training of teachers

The implications of this article for the training of secondary-level teachers of mathematics are rather obvious it seems. Many teacher-training institutions are currently offering courses in the mathematical theory of probability and statistics, and some are offering further work in statistical techniques. If the mathematics departments in our high schools and colleges are to meet the future demands for persons trained in this field, I think that all of the institutions preparing teachers of mathematics will need to include offerings in probability and statistics in their curriculum. A course in the mathematical theory of probability and statistics would seem a minimum, but further offerings including a course in statistical inference are also desirable. Furthermore, I feel that these institutions will need to include courses in this area in their summer-school programs to help prepare teachers to take their places in the secondary-school mathematics programs of the second half of the twentieth century. Substantial support for this point of view can be cited from the 1958 report of the Midwest Regional Conference on Science and Mathematics Teacher Education [9] and the report of the Secondary School Curriculum Committee. [12]

The challenge to educators and administrators, as I see it, is to come to some agreement as to how, when, and where probability and statistics are to be taught to allow us to fulfill best this unfulfilled mathematical need and to supply the trained personnel and facilities to do it. In this vein, I should like to interject an interesting observation as a final thought. In my readings on this subject, I found no one who said that probability and statistics should *not* be taught somewhere in the senior high school or junior college programs. I do not know whether this is because of the recent popularity of the field or because the "yeas" are heard from but not the "nays"; perhaps this is one of our future realms of consideration.

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Statistics for selected secondary-school students

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A description of a course of study for a one-semester course in statistics.

DURING THE PAST DECADE a revival of interest in the curriculum in secondary-school mathematics has been exhibited by mathematicians and members of various mathematical associations. In this period of study and redesign, one of the areas of mathematics receiving some attention is that of the statistical analysis of data.

In many instances curriculum revision has been attempted through uncontrolled judgment or opinion, leading to an accretion of courses in the secondary-school program. Before new subject matter is interjected into the curriculum, some knowledge of the limits of achievement for apt secondary-school students is necessary.

In accordance with this concept, a course was formulated and then taught to a group of apt students. The students were considered representative of the more proficient secondary-school mathematics students. The degree of comprehension was assessed by a testing instrument composed of items cast primarily in the direction of determining the degree of understanding, as well as computational skills.

This paper is concerned primarily with a description of the course of study followed during this one-semester period of instruction. Table 1 presents the major subject matter areas and the amount of time expended on each.

TABLE 1

TIME SCHEDULE FOR PRESENTATION OF THE AREAS OF SUBJECT MATTER

Area	Time in Hours
1. The components of a statistical investigation	7
2. Measures of central tendency and dispersion	7
3. Elementary probability	10
4. Discrete distribution functions	12
5. The normal distribution	7
6. Tests of hypotheses	30
7. Developing the statistical report	5
Total	80

The first seven hours of the course were spent discussing various components of a statistical investigation—continuous, discrete, qualitative, and quantitative variables; population and sample; types of descriptive techniques, including histograms, frequency polygons, cumulative-frequency polygons, and ogives. Distinction was made between a continuous distribution and a discrete distribution.

The second major area was concerned with the measures of central tendency and dispersion. In the discussion of the measures of central tendency—mean, median, mode—these factors were presented as one means of locating a distribution within the continuum of the number system. The measures of dispersion—variance and standard deviation—were approached through the use of the *least-squares technique*. The discussion of this material was limited primarily by the mathematical background of the students, notably the lack of the calculus. The measures of dispersion were presented as further ways of describing a distribution of variables. When appropriate, students were told to code data for ease in computation.

As an introduction to types of frequency distributions, it was thought necessary to present some of the fundamentals

of elementary probability. Within this area these concepts were developed:

1. The probability of the occurrence of a single event.
2. The idea of sample space.
3. The addition rule.
4. The multiplication rule.
5. Combinations.
6. Permutations.

These ideas were given an empirical basis to the extent that dice, coins, playing cards, and items of like nature were used as sources of data.

Following this discussion of some of the simpler forms of probability the discrete distribution functions—binomial, Poisson, hypergeometric—were discussed. The conditions under which each of these functions was applicable were pointed out. It was explained to the students that the binomial distribution is the basic distribution and that the other two discrete distributions are approximations of this function.

The normal population and the normal distribution were presented next. The normal distributions of separate values and of sample means were treated. The students were asked to carry out certain exercises related to determining the probability of the occurrence of a range of values as well as a range of sample means in a given normal distribution.

With the distribution of sample means, it was appropriate to consider the *central-limit theorem* and the standard error of estimate. Empirically this was accomplished by taking a population of poker chips numbered according to a specific normal distribution and comparing the distribution of the single values with distributions of sample means for samples of varying sizes. The relationship of the variances of the single value and sample-mean distributions tended to implement the discussion of the standard error of estimate. Also in connection with the distribution of sample means, the distribution of sample proportions was begun.

Subsequent to this preliminary work, the students were given their initial introduction to the concept of statistical inference. The thought of making a statement concerning the probability of the occurrence of a sample mean within a given hypothesized population was developed. In this case, both of the population parameters (mean and standard deviation) were known. Included within the framework of this section were the concepts of point estimates of the true population mean, confidence-interval estimates of the true population mean, and sample size necessary for a prescribed degree of accuracy in estimating the mean. The Type II or β error and the power of the test $(1-\beta)$ were also discussed. The students were given a format for setting up the solution of problems involving the testing of hypotheses:

1. Statement of the problem or question of concern.
2. Statement of the statistical null hypothesis.
3. Statement of the level of significance and the placement of this probability.
4. Statement of the critical regions.
5. Computation of the sample statistic.
6. Action with respect to the test hypothesis (i.e., accept or reject the hypothesis on the basis of the data). When the hypothesis was rejected, the student was asked to calculate the appropriate confidence-interval estimate of the true mean for the population from which the sample was drawn.

The testing of hypotheses for single population proportions or percentages was also discussed at this time. It was pointed out that industrial testing is often done in terms of defective parts or merchandise and that this technique is used primarily in these instances.

The students were then presented with the more prevalent situation in which the population standard-deviation parameter is not known. The idea of using an estimator, the sample standard deviation, in the test of the hypothesis led to a discussion

of student's t-distribution. The associated confidence-interval estimate was also presented.

After completion of the discussion pertaining to the testing of hypotheses for single population parameters, the next step was to present the methods for testing hypotheses concerning the equality of two population parameters. The tests for equality of means for the normal distribution with the population standard deviations given and with the population standard deviations not given were the same, with slight modification of one instance. Where a common variance could be assumed in the t-test, the pooled-variance concept was presented. The general case for this was presented, since essentially the same type of statistic would be used in the later subject of the analysis of variance. The testing of hypotheses for paired or correlated data was also presented at this time.

Since the chi-square distribution was used in testing hypotheses concerning the equivalence of two or more population proportions as well as in other tests, discussion of some of the characteristics of this distribution was appropriate. Associated with this preliminary discussion, certain exercises in the empirical construction of the distribution were carried out. A box of poker chips, each chip numbered in accordance with a normalized distribution, was used in obtaining samples of size n. Samples of four chips were drawn until one hundred such samples had been accumulated. One hundred samples of size two were also drawn. The sample variance of each of the samples was computed and then displayed as histograms for sample variances of size two and size four.

In addition to these, the chi-square distribution or a modification thereof was used in tests for independence of data, tests for goodness-of-fit, and tests of hypotheses concerning a single variance and the estimation of the population variance. The latter test was shown to be based upon the distribution of χ^2/d.f.

To test hypotheses concerning the equality of two population variances and the equality of more than two population means, it was necessary to promote an understanding of the F-distribution. It was further explained that when tests concerning the equality of two population means were to be carried out and the assumption of the common variance could not be made, it would be mandatory to use the F-test for equality of variances.

The analysis of variance technique was presented as a way to test hypotheses concerning the equality of more than two population means. It was pointed out to the students that certain basic assumptions were necessary before this particular technique was applicable (i.e., common variance within the populations under consideration, normality of distributions, and randomness of samples). Both the one and two fixed-factor classification procedures were examined.

In the approach to the last section of the course, it was felt that a review of the mathematics involved in determining the equation of a straight line from observed data would be appropriate. With this background, discussion proceeded to the computation of the regression line for sample data. At this point the *method of least squares* was considered again. This latter concept was used in developing the method of the *product-moment coefficient of correlation*. Tests for hypotheses concerning (1) the slope of the regression line as an indicator of dependency of data, (2) linearity of data, (3) ρ being equal to zero, (4) ρ being some value other than zero (using the Fisher z-transformation), and (5) confidence interval estimates for $\mu_{y.x}$, B, A, X, Y_c, $\sigma^2_{y.x}$, Y, and ρ were included.

Careful consideration of the test results as well as evaluation of statistical research projects carried out by each student showed that apt secondary-school students are able to apply principles of statistical inference.

An evaluation of a secondary-school mathematics program for able pupils

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What one school system did in an attempt to evaluate the results of its experimental program upon students and teachers.

THE KANAWHA COUNTY, West Virginia school system instituted at the beginning of the 1959–60 school year a pilot mathematics program designed for able mathematics pupils in four senior high schools and their attendant junior high schools. Previously in these schools and in the other secondary schools in the county, pupils began the study of algebra in the ninth grade and instruction in mathematics, for even the most able, terminated with one-half year of solid geometry and one-half year of trigonometry offered in the twelfth grade. Plans for the pilot program were developed during the 1957–58 and 1958–59 school years. A committee of teachers and principals developed the sequence of courses and the content and sequence of topics to be taught in the various courses. Recommendations of the National Council of Teachers of Mathematics, the College Entrance Examination Board Commission on Mathematics, and materials coming from a number of mathematics study groups were studied. The sequence of courses finally adopted is shown in the table below:

SEQUENCE OF COURSES FOLLOWED IN THE PILOT MATHEMATICS PROGRAM

Grade	Course
7	Accelerated Arithmetic
8	Elementary Algebra
9	Intermediate Algebra
10	Combined Plane and Solid Geometry
11	Trigonometry and College Algebra (one-half year of each)
12	Mathematical Analysis

A course of study, describing the topics to be taught, was published in mimeograph form prior to the opening of the 1959–60 school year.

The seventh-grade arithmetic course is designed to include material previously taught in the county in the seventh- and eighth-grade arithmetic courses. The course is planned to give students an understanding of our number system, the application of the fundamental operations to problem-solving in household mathematics, business practice and measurement, and to introduce the students to the study of algebra and geometry. Some of the vocabulary of "modern mathematics," particularly the notion of sets, is introduced in this grade.

The eighth-grade algebra course is similar to the elementary algebra course usually taught at the ninth-grade level. The vocabulary and concepts of "modern mathematics" are extended. Emphasis is placed on the idea of leading pupils to develop concepts through meaningful mathematical experiences. The concept of proof, both algebraic and geometric, which is introduced in the seventh grade, is extended. The course covers the usual topics through quadratic equations. Some additional topics, such as descriptive statistics and numerical trigonometry of the right triangle, have been added.

The ninth-grade intermediate algebra course is also similar to the intermediate algebra course usually offered in the eleventh or twelfth grade. Again meaning

is stressed. Concepts are developed and extended by the student through meaningful experiences. Students are trained to discover "why?" Algebraic proofs are developed. The idea of the set as a unifying concept is emphasized. Topics taught include irrational numbers, exponents, logarithms, imaginary and complex numbers, systems of equations, polynominal functions, trigonometry, progressions and series and permutations, combinations and probability.

The integrated course in plane and solid geometry taught in the tenth grade is based on the assumption that the essential concepts in solid geometry can be taught best by relating them to similar concepts from plane geometry. The student learns the relationship existing between two perpendicular planes, for example, as an extension of the relationships existing between two perpendicular lines. To quote from the Course of Study, "one of the goals of the integrated geometry course is, to enable the student to visualize Euclidean geometry, plane and solid, as a mathematical system serving as an effective model of the physical universe."

The course in trigonometry is designed to occupy one semester in the eleventh grade. A strong emphasis is placed on the theoretical aspects of the subject. Topics are covered in such a way as to extend the student's knowledge of mathematics as a logical structure, rather than making the course one in "plane surveying."

The college algebra course, which occupies the second half of the eleventh grade, is designed to approximate a course by the same title which would be offered to first- or second-year college students. The textbook chosen for the course is one that is currently being used as a text in many colleges in West Virginia. Topics taught include theory of equations, inequalities, complex numbers, progressions, infinite series, variations, permutations, combinations and probability, statistics, and limits.

The course in mathematical analysis taught in the twelfth grade is a combination of analytic geometry and differential and integral calculus. This course is not designed to replace a college calculus course but to prepare students adequately to begin their college mathematics program at the calculus level.

In order to permit pupils beyond the seventh-grade level to enter the program, the planning committee provided for able students to enter the program at the tenth-grade level by taking both the integrated geometry course and intermediate algebra and for twelfth-grade pupils to take the mathematical analysis course as an additional subject.

The following criteria were adopted to be used as the bases for placement of pupils in the program:

1. The pupil and his parents desire such placement.
2. The pupil has demonstrated a high level of academic achievement.
3. The pupil has demonstrated a keen interest in mathematics.
4. The pupil has received scores on group tests of scholastic ability which place him or her within the upper 25 per cent of the school population at his grade level or has I.Q. scores of 115 or above.

Pupils are scheduled into these classes individually. They are permitted to drop out of the program if they find the classes too difficult. Pupils who plan to continue their education beyond high school in technical or scientific fields and who meet the established criteria are particularly encouraged to enroll in the program. All students who complete the program are encouraged to take the CEEB Advanced Placement Examination in mathematics.

During the 1958–59 school year, in connection with the planning for the program, a limited number of classes were started, and some preliminary evaluation was made. Four county junior high schools, Horace Mann, Lincoln, Thomas Jefferson, and Woodrow Wilson taught sections

of seventh-grade accelerated arithmetic classes. Nitro High School, a six-year high school, taught elementary algebra to one section of eighth-grade pupils. Charleston High School taught, as an additional offering, mathematical analysis to a group of twelfth-grade pupils who were taking regularly scheduled classes in trigonometry and solid geometry.

The California Arithmetic Test was administered to the seventh-grade pupils in these accelerated arithmetic sections at the end of the 1958–59 school year. The pupils tested generally scored well above the end-of-year norms established by the publisher for eighth-grade pupils. At the end of the 1958–59 school year, the Cooperative Elementary Algebra Test was given to the pupils taking algebra in the eighth grade at Nitro and also to selected sections of ninth-grade pupils taking elementary algebra at the same school. Eighth- and ninth-grade pupils were matched according to I.Q. scores, and the algebra test results were compared. No significant differences were found between the scores of the eighth-grade pupils and those of the matched ninth-grade pupils. The test results showed that both groups achieved well above the national norm, established by the publisher for ninth-grade pupils who have completed one year of study. Information secured by the teachers at Charleston High School during the 1959–60 school year showed that of the seventeen pupils enrolled in the class in mathematical analysis the previous year, about half had secured advanced placement in college mathematics courses, either as a result of their scores on the CEEB Advanced Placement Examination or from entrance examinations given at the different colleges and universities in which these pupils had subsequently enrolled.

At Nitro High School an intensive course in solid geometry was taught during the first eight weeks of the 1959–60 school term to an accelerated group of eleventh- and twelfth-grade pupils, who had already completed plane geometry and intermediate algebra. Upon completion of the solid geometry course, this class then began the study of analytic geometry. This arrangement was expedient in providing an enriched program for these pupils, who otherwise would not have had the opportunity to take additional mathematics courses in high school. At the end of the eight-week period, the Cooperative Solid Geometry Test was administered. The percentile equivalent of the median score for the group, based on the publisher's norms for eleventh-grade pupils who had completed one-half year of study of solid geometry, was ninety-four, with the percentile ranging from thirty-six to ninety-nine. The teacher and principal felt that the results showed that able pupils such as these could complete the study of solid geometry in half or less than half of the usual time (one semester) that is normally devoted to the course.

Plans were made to evaluate the county pilot program at the end of the 1959–60 school year in all the participating schools. A questionnaire was developed to be given to pupils in the program and another to be given to teachers and principals. The questionnaires were administered at all the schools participating in the program to all pupils enrolled in the pilot program, to all mathematics teachers teaching in the program, and to the school principals in May, 1960. Usable results were received from a total of 607 pupils and from 22 teachers and principals. The questions asked on the questionnaires and a summary of the results obtained are shown in Table 1 and Table 2.

Teachers and principals were asked also to make comments concerning the program. No suggestion was given concerning the nature of the comments desired or the subjects on which the respondent might comment. A study of the comments received appears to show that most teachers found that teaching these classes required more time and effort for preparation than they had been accustomed to

TABLE 1

RESULTS FROM QUESTIONNAIRES ADMINISTERED TO KANAWHA COUNTY PUPILS PLACED IN THE PILOT MATHEMATICS PROGRAM

Questions asked	Number of responses
1. How do you feel about mathematics classes of this kind?	
a) I like this kind of class better than a regular class.	506
b) I neither like nor dislike this class more than a regular class.	91
c) I dislike this kind of mathematics class.	7
d) No response.	1
2. How has what you learned this year compared with what you think you would have learned in a regular class?	
a) I have learned more.	520
b) I notice no difference.	14
c) I think I would have learned more in a regular class.	14
d) No response.	5
e) If I understood, I would have learned more.*	1
3. How does the understanding you have gained in your class this year compare with that of the former mathematics classes you have taken?	
a) I have understood the things we have had in this class better than in former classes.	299
b) I notice no difference in the meaningfulness of the things in this class than in former classes.	157
c) I have more difficulty understanding the things that are taught in this class.	133
d) No response.	4
4. How have your friends who were not placed in the pilot class felt about your being in the class?	
a) They accept my placement in the class and are happy for me.	178
b) They don't seem to be concerned about it.	402
c) Some of them resent my being in the class.	25
d) No response.	2
5. How do your parents feel about your being placed in the class?	
a) They are happy about it.	509
b) They haven't said much about it one way or the other.	86
c) They do not like my being in the class.	6
d) No response.	1
6. How have you felt about the work that has been required in the class this year?	
a) There has been too much work.	65
b) I notice no difference from the work required in regular mathematics classes.	348
c) I have had to do less work in this class than I would have had in a regular class.	31
d) No response.	32
e) More work but not too much.*	122
f) Not enough work.*	2

* Choices "*e*" and "*f*" represent responses which were written in by students.

TABLE 2

RESULTS FROM QUESTIONNAIRES ADMINISTERED TO KANAWHA COUNTY MATHEMATICS TEACHERS AND PRINCIPALS

Questions asked	Number of responses
1. How have you found the accelerated mathematics program to function at your school?	
a) Satisfactorily.	22
b) Unsatisfactorily.	0
c) No response.	0
2. How do you feel that students in these classes progressed in comparison with the progress they would likely have made in regular classes?	
a) Better.	21
b) No noticeable difference.	1
c) Poorer.	0
d) No response.	0
3. What do you think about the appropriateness of the content of the courses in the program for the pupils who enroll in the program?	
a) More appropriate to pupils' needs and interests.	19

TABLE 2 (continued)

Questions asked	Number of responses
b) No noticeable difference.	0
c) Less appropriate to pupils' needs and interests.	0
d) No response.	3
4. What is the most prevalent student attitude toward placement in the classes?	
a) Desirable.	21
b) Pupils indifferent as to being placed in these classes or in the regular mathematics program.	0
c) Pupils have to be forced to enroll in them.	0
d) No response.	1
5. What have you found concerning the attitudes toward the study of mathematics on the part of the students in the program?	
a) Better.	16
b) No noticeable difference.	4
c) Poorer.	0
d) No response.	2
6. What has been the most prevalent attitude of the pupils who have been placed in the program toward fellow pupils who have continued in the regular program?	
a) Feeling of superiority or patronization.	2
b) No noticeable change from previous attitudes toward same pupils.	16
c) Improvement in interpersonal relations.	3
d) No response.	1
7. What has been the most prevalent attitudes expressed by pupils not in the program toward fellow pupils who have been placed in the program?	
a) Favorable acceptance of this placement.	15
b) Indifference.	5
c) Disapproval of such placement.	0
d) No response.	2
8. What effect has placement in the program had on the attitude of the pupils so placed toward school generally?	
a) Pupils' attitudes have improved.	7
b) No noticeable difference.	13
c) Pupils' attitudes have deteriorated.	0
d) No response.	2
9. What have you found to be the most prevalent attitudes of parents toward having their children placed in the program?	
a) Parents are favorably inclined.	19
b) Parents are indifferent or willing to leave placement to the judgment of the school.	2
c) Parents are reluctant to have their children placed in the program.	0
d) No response.	1
10. What have you found regarding teaching classes in the program?	
a) Classes are easier to teach.	11
b) I notice no difference.	2
c) Classes are harder to teach.	4
d) No response.	5
11. What is your belief concerning the content of accelerated mathematics courses?	
a) Emphasis should be placed on learning more advanced concepts.	3
b) Emphasis should be placed on developing better understanding of concepts in the regular mathematics sequence.	9
c) Emphasis should be placed on both *a* and *b*.	7
d) No response.	3

devoting to previous classes. However, the teachers who made this comment also said that they had found the experience of teaching the classes to be enjoyable and rewarding. Several teachers commented that student interest in the classes was high and that pupils were self-motivated to learn. None of the teachers or principals responding made any comment about the program which could be interpreted as adverse criticism.

In addition to the questionnaires which were administered, standardized tests of arithmetic, elementary and intermediate algebra, and plane and solid geometry were also given in May, 1960, to sections of pupils in the program. The California Arithmetic Test, intermediate tests, Forms AA and BB, was administered to the seventh-grade accelerated arithmetic sections. The Cooperative Elementary Algebra Test was administered to the

eighth-grade algebra sections and the Cooperative Intermediate Algebra Test was given to pupils completing intermediate algebra in grade nine. Tenth-grade pupils in some integrated sections were given both the Cooperative Plane and Cooperative Solid Geometry Tests. Others were given only the plane geometry test. The tenth-grade pupils who took the solid geometry test were taking integrated geometry in addition to intermediate algebra. Tenth-grade pupils who took only the plane geometry test included also sections of pupils who were taking integrated geometry as their only mathematics course at this grade level. There are no known standardized tests which are appropriate for the classes in mathematical analysis.

Since the program has been in operation for only one year, there were no classes in trigonometry or college algebra which were a part of the pilot program.

Since all tests were administered at the end of the first full year of the program's operation, a majority of the pupils tested were in the seventh-grade sections. Pupils tested in the eighth grade were the pupils at the specified junior high schools which had taught seventh-grade accelerated arithmetic in 1958–59. Similarly, the ninth-grade pupils who took the intermediate algebra test were the pupils at Nitro who had been tested at the end of the preceding year and matched with a ninth-grade control group.

Table 3 shows the results from the Cali-

TABLE 3

RESULTS FROM THE CALIFORNIA ARITHMETIC TESTS GIVEN TO SEVENTH-GRADE ACCELERATED ARITHMETIC PUPILS AT HORACE MANN, LINCOLN, THOMAS JEFFERSON, AND WOODROW WILSON JUNIOR HIGH SCHOOLS

Grade equivalent	Number of pupils			Grade equivalent	Number of pupils		
	Arithmetic reasoning	Arithmetic fundamentals	Arithmetic total		Arithmetic reasoning	Arithmetic fundamentals	Arithmetic total
5.0		1		8.4		2	
5.4		1		8.5	2	3	4
5.8		2		8.6	1	4	2
5.9		1		8.7		4	
6.0			1	8.8	3	3	2
6.1			1	8.9	3		3
6.2		1		9.0	1	2	2
6.4		1	1	9.1	2		
6.5	1	1	1	9.2		8	3
6.6	1	2	1	9.3	8	5	2
6.7		1		9.4		6	9
6.8		1		9.5	3	4	8
6.9		3		9.6	5		4
7.0		1		9.7		11	7
7.1		1		9.8	7	8	7
7.3		3	1	9.9	10		1
7.4		4		10.0	6	7	8
7.5	1	4	1	10.1			2
7.6		4	2	10.2	16	5	4
7.7		1	1	10.3			2
7.8			2	10.4	5	17	1
7.9		2	4	10.5	10		2
8.0	1	1	3	10.6		3	11
8.1	1			10.7	9	1	7
8.2		2	3	10.8	10	6	4
8.3	3	3	5	Above 10.8	74	43	61
Median grade equivalent					10.7	9.8	10.2
Total number of pupils tested					183	183	183

TABLE 4

RESULTS FROM THE COOPERATIVE HIGH SCHOOL MATHEMATICS TESTS GIVEN TO PUPILS IN THE KANAWHA COUNTY PILOT MATHEMATICS PROGRAM

Test	Grade level	Number of pupils tested	Range of scores		Median score	
			Standard score	Percentile	Standard score	Percentile
Cooperative Elementary Algebra Test	8th	259	84–30	99–1	63	86
Cooperative Intermediate Algebra Test	9th	39	71–46	94–10	63	74
Cooperative Plane Geometry Test	10th	170	74–24	98–1	57	62
Cooperative Solid Geometry Test	10th	60	82–50	99–21	65	81

fornia Arithmetic Tests. The grade equivalents are based on the publisher's norms.

Table 4 shows the results from the Cooperative Tests administered to the eighth-, ninth-, and tenth-grade pupils in the program. The percentiles shown are based upon the publisher's norms for end-of-year achievement. However, the results for the eighth-grade elementary algebra test are based on publisher's norms developed for ninth-grade pupils and the ninth-grade intermediate algebra test results are based on norms for eleventh-grade pupils, since these are the grade levels at which these courses are usually taught.

Discussion of the results of this survey must be divided into two categories: the results from the questionnaires and the standardized test results. The former indicates the reception which this pilot program has been given by students, teachers, and principals, and, indirectly, by the parents of the students in the program and by other pupils who are not in the program. The latter show the level of achievement.

The results from the questionnaires show that this pilot program has been accepted and is favored by almost everyone concerned. A great majority of the students questioned reported that they liked these special classes better than the mathematics classes in which they had been previously enrolled and that their parents liked having them in the classes. The students also said that they had learned more in the classes, although a number of pupils said that they had been required to work harder. It is interesting to note that a large per cent of the pupils questioned wrote in the response that they had to do more work but not too much.

Teachers unanimously reported that they had found the program to be satisfactory and were almost unanimous in saying that they felt that students had made more progress in these special classes than they would have made in the regular mathematics program. The teachers said also that the students' attitudes had improved and that their parents had reacted favorably to the classes. Generally, the responses of the teachers, all of whom have had one or more years' experience in the program, appear to be a vote of confidence for the program. The fact that the teachers who responded to the questionnaire were, for the most part, the same teachers who planned the program admits the possibility, however, that these teachers identify with the program and are not able to view it objectively.

Some of the teachers helping with the planning of the program were concerned with the effect that placement in a special class, such as one of the classes in the pilot program, might have on the pupils' interpersonal relations. Questions number 4 on the pupils' forms and numbers 6 and 7 on the teachers' questionnaire were designed to elicit information concerning this

point. Teachers reported that the attitudes of pupils placed in the program toward their fellow pupils who were not so placed was not noticeably affected. The teachers also thought that pupils not in the program accepted the program and did not appear to resent other pupils being placed in the group.

The responses showed that the pupils in the program felt that their friends were, in the main, either happy that they had been placed in the program or were unconcerned. It should be pointed out here, however, that pupils not in the program were not questioned as to their attitudes toward the program. This represents a weakness in the plan for the survey. Less than 5 per cent of the pupils placed in the special class reported that their friends resented their having received such placement. One student wrote in a comment on this question that may be significant. He or she said, "Most of my friends are also in this class."

The Cooperative Tests chosen to be used in the evaluation appeared to be among the most appropriate standardized instruments for measuring the achievement of classes such as those in this pilot program. The California Arithmetic Test also appeared to be an appropriate instrument, being well standardized and yielding scores both in arithmetic reasoning and arithmetic fundamentals. The level of the California test chosen to be given to these seventh-grade pupils also appeared appropriate, since the test is designed for use with seventh-, eighth-, and ninth-grade pupils. However, the results showed that the California test used was not appropriate since the scores of about one-third of the pupils tested were so high that they were above the maximum grade equivalent of 10.8 school years set by the publisher. Thus, while the median grade equivalents were not affected, both the ranges of the scores and the means, if they had been desired, would not have been usable. When corresponding groups of seventh-grade pupils are to be tested again, a test having a considerably higher level of difficulty will be chosen.

The results from the seventh-grade tests showed the median scores on the two subtests and the composite score to range from two to nearly three years above the national norm. The pupils scored higher on arithmetic reasoning than on arithmetic fundamentals. More than one-third of the pupils' scores placed them three years or more above national norms in arithmetic reasoning. About one-fourth had similar results for arithmetic fundamentals and exactly one-third for total arithmetic.

Interpretation of test results from groups of able pupils such as these can only be made in terms of levels of expectation. Another weakness of this survey, from the aspect of its quality as a study, is the lack of a control group. The survey did not yield data that would show the level of achievement of pupils of comparable ability who were in the regular county mathematics program. The *Publisher's Manual* for the California test does give adjusted norms for pupils of various ability levels. According to the manual, pupils having I.Q.'s of 115 (the estimated lower limit for the classes) should receive scores on the total arithmetic test at the seventh-grade level, yielding grade equivalents 1.25 years above the national norm. This compares with a median grade equivalent for the pupils tested which is 2.3 years above the norm.

The fact that the median test scores for these seventh-grade arithmetic sections were at the tenth-grade level appears to have considerable implications for curriculum planning, grouping for instruction, and choice of instructional materials in the county. If sections of seventh-grade arithmetic classes can be formed whose median score on an achievement test at the end of the year is almost three years ahead of the national norm, then considerable study appears to be called for concerning the adequacy of the regular elementary school arithmetic program for

these students. The levels of reading vocabulary and mathematical concepts contained in the textbooks used by these pupils and even the adequacy of the pupils' program in the sciences and other subjects should be re-evaluated.

The results from the Cooperative Algebra Tests, both for the elementary and intermediate classes, showed that the students tested made very high scores on both tests. The national median score is, of course, the fiftieth percentile. Results for the elementary algebra students were equivalent to the eighty-sixth percentile, based on national ninth-grade norms. The ninth-grade pupils taking intermediate algebra also scored well above the national norms which are based on tests administered to eleventh-grade pupils. It must be concluded from the results of this study both that these Kanawha County pupils have demonstrated superior achievement in algebra and that elementary algebra can be taught to eighth-grade pupils and intermediate algebra to ninth, if the pupils are selected according to criteria similar to those used in this county.

The results from the Cooperative Geometry Tests do not appear to be as incisive as the results from the arithmetic and algebra tests. The county pilot mathematics program will require five years from its inception to be fully instituted in all grades. However, the test results show that pupils in the geometry sections tested achieved above the national norm, and that the achievement of many pupils was definitely superior. The special solid geometry test given earlier in the school year at Nitro appears to have demonstrated that solid geometry can be taught successfully in half the time which has been allocated to it in the past. The solid geometry test results for tenth-grade students indicate that solid geometry can be taught successfully at a lower grade level than it has been placed previously in the county.

Summary and conclusions

A pilot program in mathematics designed to allow able students to proceed through a mathematical sequence culminating in mathematical analysis in the twelfth grade was instituted in four Kanawha County high schools and their attendant junior high schools at the beginning of the 1959–60 school year. An evaluation of the results obtained was made at the close of the school year. The evaluation took the form of questionnaires administered to pupils, mathematics teachers, and principals of the schools concerned and standardized achievement tests administered to pupils in the program.

The results from the questionnaires showed that almost all teachers and a large majority of pupils were pleased with the pilot program. Students reported that they felt that they had learned more in these special classes than in their previous classes. Teachers reported that they also felt the pupils had learned more and that there had been a general improvement in the students' attitudes.

Standardized achievement test results showed the seventh-grade arithmetic pupils in the program to be almost three years ahead of national norms. Eighth-grade pupils completing first-year algebra were found to be well above the national norms for ninth-grade algebra students. Similarly, ninth-grade pupils taking intermediate algebra were found to be well above the national norms for eleventh-grade pupils. Achievement tests in plane and solid geometry also showed pupils to be above national norms. While the test results generally indicated that the pilot program is proving to be successful, comparisons were not made of the results obtained in these selected sections with results obtained in the regular mathematics sequence existing in other schools in the county. Nor were the test results evaluated on terms of levels of expectation.

Some implications of programmed instruction for the teaching of mathematics

RALPH T. HEIMER, *Pennsylvania State University, University Park, Pennsylvania.*

Can and should mathematics be taught by the use of programmed-instruction materials and machines?

THE ALERT TEACHER of mathematics will soon be occupied by what appears to be another significant development in the area of mathematics education. In addition to being confronted with a deluge of proposals for curricular change, the astute teacher may soon find the specter of automation forcing him to subject his mode of instruction to critical evaluation. Reference is being made to the advent of an entirely new concept in the theory of learning and teaching—a concept of "automated instruction."

"Automated instruction" is a phrase which refers to an instructorless method of teaching or learning and represents in some sense—ideally, at least—the classical process of tutoring. In recent months, there has been an eruption of interest in the technique of automated instruction; in particular, a great deal of interest has been manifested in the preparation of mathematical materials for this type of instruction. This has not been a chance event. Mathematics seems unusually well suited for such treatment because of its high degree of logical organization, and it is in the interest of keeping the high school mathematics teacher informed of what may well become a significant aspect of the teaching of mathematics that this article is written.

It would, perhaps, be more accurate to refer to the subject of automated instruction as "programmed instruction," since the most important feature of the method involves the "programming" of a body of knowledge for consumption by a group of learners. This process of programming amounts to taking the body of material to be learned and breaking it into an orderly sequence of units. Each unit is, in turn, logically organized into small steps which are normally framed as questions.

The character of the actual learning situation varies with the mode of programming employed, and there are at the present time two basically different techniques in wide use. One method consists of asking the student to answer a series of questions in a carefully organized sequence. After the student has made an overt response to a question, he is provided with the correct answer. In the writing of such a program, it becomes important to attempt to present new information in relatively small steps, so that if an incorrect response should by chance be given, the mere expedient of showing the student the correct answer suffices to "set him straight." The following is an example of a sequence of such items:

1. It is sometimes useful to combine two sets to form a new set called their *union*.

   For example, the *union* of the sets $\{1,2\}$ and $\{3,4\}$ is $\{1,2,3,4\}$.

What is the *union* of the sets $\{1,3\}$ and $\{2,4\}$?

2. $\{1,2,3,4\}$.

Suppose $A = \{1\}$ and $B = \{2\}$. The *union* of sets A and B is the set ____.

3. $\{1, 2\}$.

The set $\{2.14, 6, \frac{1}{2}, -2\}$ is said to be the _____ of the sets $\{6, -2\}$ and $\{\frac{1}{2}, 2.14\}$

4. union.

The *intersection* of $\{0, *, 7, \square, .5\}$ and $\{7, \square, 2, +\}$ is _____.

5. $\{7, \square\}$.

The *union* of $\{0, *, 7, \square, .5\}$ and $\{7, \square, 2, +\}$ is _____.

6. $\{0, *, 7, \square, .5, 2, +\}$.

The union of the set of odd numbers and the set of even numbers is the set _____ .

7. $\{1, 2, 3, 4, \cdots\}$.

Etc.

(It is to be noticed that the answer to any given question appears in the subsequent frame.)

The second method differs from the first primarily in one respect: this technique of programming provides the learner an additional degree of freedom. The steps are usually larger, and the student must select a response from a group of possibilities. When an incorrect response is given, the learner is directed to a series of items (questions) through which an attempt is made to eliminate the difficulty that led to the incorrect response. The more alert student, of course, bypasses these items and continues working in the main stream of the program.

Programmed materials can be presented to students in various ways. There are available many so-called "teaching machines" which have been designed for this purpose. A more recent development is the programmed textbook which is written in a manner whereby the text itself functions as a machine. It should be emphasized, however, that the central values of automated instruction lie with the program, not with the mode of presentation. These values are as follows:

1. The student is continuously involved in the learning process. He must make overt responses to the questions which form the program in order to proceed.
2. The student gains immediate reinforcement; that is, the student is constantly kept informed of the correctness of his responses.
3. Each student can proceed at his own rate of speed. Thus, individualized instruction becomes a reality.

These advantages of programmed instruction are recognized as highly desirable factors in the learning situation and are responsible in large measure for the great deal of activity and enthusiasm being exhibited in this area.

There are a number of mathematics programs currently available in one form or another, and many others are in the process of being written. These include elementary algebra, plane geometry, intermediate algebra, trigonometry, analytic geometry, elementary calculus, differential equations, and statistics. At Pennsylvania State University we have embarked upon a research project under the provisions of a Title VII grant in which we are trying to adapt, via closed-circuit television, principles of programmed learning to large-group instruction. The closed-circuit television setup will function as our "teaching machine," and this may result in an entirely new concept of the usefulness of television in an educational setting. We are also preparing our own materials, and the writer is carrying a central responsibility in the development of a program in basic college mathematics which will have a strong algebraic orientation.

There is a great deal to be learned about programmed instruction. For instance, in what capacity is it most useful? Just how effective is it? In what areas will it find its fullest application? The answers to these questions and many like them will be forthcoming and may well demand more than just passing interest.

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Letters to the editor

Dear Editor:

I was reading "Problems involving overlapping finite sets" by Brother Alfred in the November, 1960, issue (pages 524–32), and felt it would serve me very well for illustrative work. However, I have run across a difficulty in one of his illustrations and would appreciate clarification.

On page 525 Brother Alfred uses as an example a small town with 7,635 male adults. These are broken down as follows:

A_1	5,492 coal miners	$N_1 = 5{,}492$
A_2	6,057 Democrats	$N_2 = 6{,}057$
A_3	4,135 affiliated with a church	$N_3 = 4{,}135$
A_{12}	4,532 coal miners and Democrats	$N_{12} = 4{,}532$
A_{23}	3,957 church affiliates and Democrats	$N_{23} = 3{,}957$
A_{13}	3,553 church affiliates and coal miners	$N_{13} = 3{,}553$
A_{123}	3,337 church affiliates, Democrats and coal miners	$N_{123} = 3{,}337$

My problem results from the fact that when we break down the church affiliates we find what appears to me to be a contradiction.

$$N_3 = 4{,}135$$

But

$$N_3 = N_{123} + N_{1\bar{2}3} + N_{\bar{1}23} + N_{\bar{1}\bar{2}3}$$

or

$$N_3 = N_{13} + N_{23} - N_{123} + N_{\bar{1}\bar{2}3}.$$

Upon substituting the figures given, we find:

$$4{,}135 = 3{,}553 + 3{,}957 - 3{,}337 + N_{\bar{1}\bar{2}3}$$

This leaves us with the result that $N_{\bar{1}\bar{2}3} = -38$, which indicates the original set of data was inconsistent. Clarification is awaited with interest.

Very truly yours,
LUCIEN K. WERNICK
Freedom Plains Road, RD#3
Poughkeepsie, New York

Editor's Note: Mr. Wernick is right. The number $N_{\bar{1}\bar{2}3}$ would be -38. The data as given are inconsistent.

Mr. Wernick wishes to credit Michael Lipshitz, a student at Bronx High School of Science, for actually having discovered the discrepancy. Mr. Wernick reports he has found a second discrepancy on pages 529–30 in the example concerning drama students and football players.

Dear Editor:

On page 644 of the December, 1960 issue of THE MATHEMATICS TEACHER is an article on the decimal approximations of π.

On page 648 of the same issue appears a table entitled "Cumulative Distribution of the First 16000 Decimal Digits of π." Under the five thousand mark, the sum of the distributive digits should equal 5000, whereas they only total 4991. Would you please correct this error in your next issue? Thanking you,

I remain,
MARTIN HOFFMAN
Student at
Boston Latin School

Editor's Note: The editor of the "Historically Speaking,—" department when notified of this error, noted that the difference between 4991 and 5000 is 9. This suggested a transposition error. The last number in the row after 5 should read 521 instead of 512.

"Science teachers, encouraged by having adequate laboratory equipment, are attending inservice training institutes and summer courses in increasing numbers."—*From "Schools in Our Democracy," Office of Education, U.S. Department of Health, Education, and Welfare.*

A Pascal pyramid for trinomial coefficients

EVELYN B. ROSENTHAL, *Fieldston School, New York, New York.*

Some students enjoy the Pascal triangle. Do they know about a Pascal pyramid?

THE CONSTRUCTION of the Pascal triangle for binomial coefficients is well known; each number in it, $\binom{n}{j}$, is the sum of the two adjacent numbers above it in the preceding row. This method of constructing the coefficients works because of the property:

$$\binom{n}{j}=\binom{n-1}{j-1}+\binom{n-1}{j}.$$

Similarly, a three-dimensional analogous construction of the trinomial coefficients can be made, based on the property:

$$\binom{n}{i,j,k}=\binom{n-1}{i-1,j,k}+\binom{n-1}{i,j-1,k}+\binom{n-1}{i,j,k-1},$$

where

$$i+j+k=n.$$

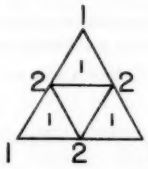
The symbol $\binom{n}{i,j,k}$ represents the coefficient of $x^iy^jz^k$ in the expansion of $(x+y+z)^n$; this number is also the number of combinations of n things taken i, j, k at a time.

The pyramid formed by these numbers, each the sum of three adjoining numbers in the layer above, is most easily represented by its layers.

The top layer, where $n=0$, is a point, the vertex of the pyramid. This point represents $(x+y+z)^0=1$. The next layer shows $n=1$:

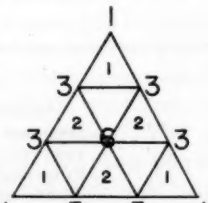
$$(x+y+z)^1=x^1+y^1+z^1.$$

For $n=2$:



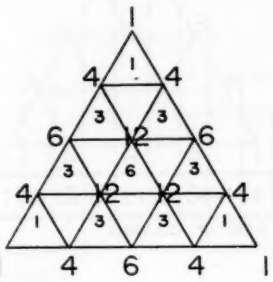
$$(x+y+z)^2=x^2+y^2+z^2+2xy+2xz+2yz.$$

For $n=3$:

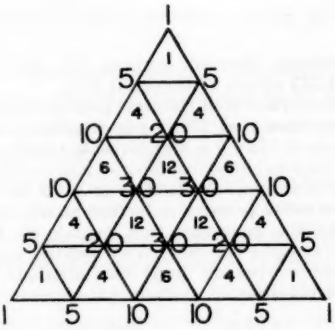


$(x+y+z)^3$ has terms of the form x^3, $3x^2y$, and $6xyz$.

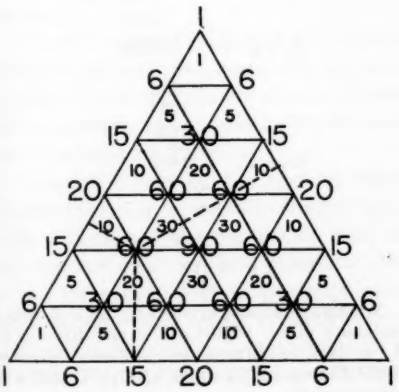
For $n=4$:



For $n=5$:



For $n=6$:



The method of construction is as follows: the large numerals at each intersection are the coefficients for the given n. To get them, insert the numerals for the coefficients for $n-1$, that is the numerals from the previous layer of the pyramid. These symbols are placed in the triangles that "point up." At each intersection inside the large triangle, there are three of these small triangles; at each intersection on the sides of the large triangle, there are two; at each vertex of the large triangle, there is one. In each case, the sum of the numbers in the small triangles that meet at any intersection is the coefficient for that intersection.

In each layer the repetition of the patterns of the binomial coefficients is obvious.

The numbers of small triangles, traversed by the perpendiculars from any intersection to the sides of the large triangle, give the exponents for that term, e.g., the dotted lines in the diagram for $n=6$ show that a term in the expansion of $(x+y+z)^6$ is

$$60x^1y^2z^3; \qquad 60=\binom{6}{1,2,3}.$$

Similarly, from the previous diagram, a term in the expansion of $(x+y+z)^5$ is $10x^0y^3z^2$, or $10y^3z^2$ and $10=\binom{5}{0,3,2}$. Another term is $30x^2y^2z^1$ and $30=\binom{5}{2,2,1}$.

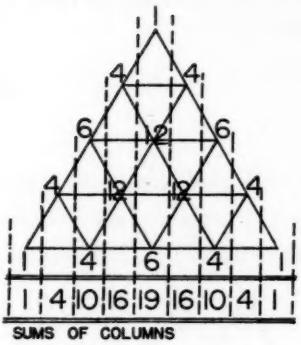
Since the sum of the perpendiculars from any point in an equilateral triangle to the sides is a constant, the sum of the exponents in any term of an expansion is also a constant.

This construction also shows that the number of terms is a "triangular number," i.e., a number of the form $\frac{x(x+1)}{2}$. Here it is $\frac{(n+1)(n+2)}{2}$, the number of partitions of n into three parts.

If we want the expansion of $(1+x+x^2)^n$, the sums of the numbers in vertical col-

umns give those coefficients. It is easily verified that if the distance (measured by the numbers of small triangles traversed) from the right side of the large triangle gives the exponent of 1, the distance from the base gives the exponent of x, and the distance from the left side gives the exponent of x^2, then all terms in the same vertical column will be like terms. For instance, using the layer for $n=4$, we find that

$$(1+x+x^2)^4=1+4x+10x^2+16x^3+19x^4 +16x^5+10x^6+4x^7+x^8.$$



Letter to the editor

Dear Editor:

This discussion is a continuation, in effect, of an article titled "A Test for Divisibility" by Sister M. Barbara Stastny, O.S.F., in the December, 1960, issue of THE MATHEMATICS TEACHER. The article develops tests for divisibility by all the primes below 100 except 83 and 97. Relatively simple tests are suggested by an extension of Stastny's work.

It will be observed that 100 is congruent to 3, modulo 97. This suggests that for all numbers of the form $100x+y$, the number will be divisible by 97 if and only if $3x+y$ is divisible by 97. The identity $100x+y=(3x+y)+(97x)$ clearly demonstrates the universality of the conjecture. Using Stastny's algorithm, 86,434,372 is tested by divisibility by 97.

$100x+y=100(864343)+72$

$3x+y$	86434372
	×3
	2593029
	+72
$3x_{-1}+y_1$	2593101
	×3
	77793
	+1
$3x_2+y_2$	77794
	×3
	2331
	+94
$3x_3+y_3$	2425
	×3
	72
	+25
	97

Ordinary division confirms the fact that 86,434,372 equals 97 times 891,076.

It is obvious that the same principle could be used to show that $100x+y$ is divisible by 83 if and only if $17x+y$ is also divisible by 83. However, the computational labor of successive multiplications by 17 is about as great as the ordinary division process. A further search reveals that 1,000 is congruent to 4, modulo 83. $1000x+y=(4x+y)+(12)(83x)$. This identity leads to a more convenient and less laborious test. 86,434,372 is tested by divisibility by 83.

$4x+y$	86434372
	×4
	345736
	+372
$4x_1+y_1$	346108
	×4
	1384
	+108
$4x_2+y_2$	1492
	×4
	4
	+492
	496

We find that 496 divided by 83 is 5 with a remainder of 81. 86,434,372 is not divisible by 83, a fact confirmed by ordinary division. 86,434,372 equals 83 times 1,041,377 with a remainder of 81.

These two tests, with Stastny's tests, provide a complete system of tests for divisibility for primes less than 100.

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*Algebra and geometry; mathematics or science?**

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For the nonspecialist in mathematics, which subject makes the greater educational contribution, algebra or geometry?

My remarks have been organized into seven topics.

1. Pure mathematics is a glorious game played by an ever-increasing, but tiny minority. I have nothing to say about it here.

2. There are two kinds of applied mathematics: the kind that helps us to understand nature, and the kind that helps us to understand ourselves.

The first kind of applied mathematics need not concern us here. There is very little disagreement about preparation for it. The student must master the manipulative aspects of geometry, algebra, calculus, differential equations—the more the better. If he happens to understand what he is learning, he will enhance his chances for success.

The second kind of applied mathematics is the important kind, since only by understanding ourselves do we stand a chance for happiness, as opposed to the hypnotic calm induced by gadgets.

This is the only kind of applied mathematics that can be usefully taught to the vast majority of the population who will not become engineers and scientists. This is the important part of the population, for it decides what music we shall hear on our local radio stations, what programs we shall watch on television, what movies we shall see, and, by the way, who shall govern us.

* A lecture given to the Pennsylvania Council of Teachers of Mathematics, April 9, 1960, E. Stroudsburg, Pennsylvania.

Is the technique for solving quadratic equations a part of this latter kind of applied mathematics? Surely not, for it is clear that no housewife, merchant, doctor, or senator will be faced with a quadratic equation to solve. The only justification for their having learned about quadratic equations lies in the possibility that there is carry-over, that knowledge of quadratic equations will purify their thought processes in all fields of endeavor. Similar remarks apply to Pythagoras' theorem.

It is the purpose of this article to explore the relative appropriateness of algebra and geometry as vehicles for the important task of tidying the minds of the general public. I will contend that at the secondary level *geometry cannot be so used and that algebra can be, if properly presented.*

The reason for this, in a nutshell, is that geometry cannot be taught as mathematics at any level below the graduate school; hence, geometry in the high school must be a branch of science. On the other hand, algebra can be taught as mathematics in the high school.

3. Let us begin by studying geometry.

In Euclid's Proposition 1, Book I we are given points A, B. With centers at A, B we draw circles with radius AB. They intersect at C and the triangle ABC is declared equilateral.

Dismissing for a moment the questions "What is *draw*?" "What is *circle*?" "What is *triangle*?" we ask the question, "Why do the circles intersect?" If we carried out the experiment, we would find that they some-

times do not, for example, let A, B be diametrically opposite each other on the earth's surface. Then each circle contains exactly one point. (Notice that I ignore the more customary objection [1]—that one circle might pass from the inside to the outside of another without intersecting it, for example, if we deal with a co-ordinate geometry using only points with rational co-ordinates. I am more interested in the possibility that one circle may lie entirely outside the other.)

Then we ask why it can be asserted that A, B, C are not collinear. (Presumably a triangle must, by definition, have non-collinear vertices.) Perhaps the noncollinearity follows from the easily proved identity $|AB| = |BC| = |CA|$. How does it follow?

It goes without saying that Euclid's axioms yield no answers.

I submit that the only way to answer these questions at the secondary level is to accept as obvious the facts from a diagram. But a diagram is a physical object made of paper and graphite, or slate and chalk, or wood, nails, and string. Hence, my submission is that geometry must be taught at this level as a branch of science. "Proof" by diagram will convince the student. Only a few persons like David Hilbert, the great mathematician, will remain unconvinced by diagrams. This method of teaching geometry will impart a valuable spatial technique and be a useful learning experience, providing *we give up telling the student that he is seeing exact reasoning.* A wholehearted adoption of this attitude will result in economy of time and space, as pointed out in one instance by W. W. Sawyer [6].

4. The alternative is to employ exact reasoning. (As indicated above, Euclid's treatment falls short of this ideal.) Beginning with undefined terms, such as point and line, we list a set of axioms which will be unproved assumptions. Finally, theorems are proved. If diagrams are drawn, they must be used only to help fix the ideas; no special property of the diagram may be used. Thus, if we wish to show that the larger of two sides of a triangle has the larger angle opposite it, it is perfectly permissible to draw an equilateral triangle, or a right triangle, or even a collection of three zigzag curves, since the reasoning will be strictly deductive and not refer to the diagram.

As an example, consider the following special case of Euclid, Proposition 7, Book I:

Proposition: *Given two points* A, B, *there exist at most two points* C, D, *such that* ACB *and* ADB *are equilateral triangles.*

This will be a theorem in the following development, which as far as I know, cannot be materially shortened.

Point is undefined: A line is a collection of points subject to the axiom; two points belong to exactly one line. Congruence is an undefined relation between pairs of points, written $\equiv$. It is assumed that $PQ \equiv PQ \equiv QP$, that if $PQ \equiv RS \equiv TU$, then $PQ \equiv TU$, and that if $PQ \equiv RS$, then $RS \equiv PQ$. Circle is defined as usual. Let *pb PQ*, standing for the perpendicular bisector, be the set of all R, such that $PR \equiv RQ$. AXIOM: *pb PQ* is a line. AXIOM: If P, Q, R lie on a line, then *pb PQ* is parallel to (i.e., does not meet) *pb QR*. AXIOM: If P, Q, R are distinct,

$$pb\ PQ \neq pb\ QR.$$

The proofs of Theorems 1 and 2, below, are trivial.

THEOREM 1: *Two lines intersect in at most one point.*

THEOREM 2: *Two different circles cannot meet in three points.* (Euclid, III 5)

Proof: Let P, Q, R lie on circles with centers A, B. Then $AP \equiv AQ$, hence, A lies on *pb PQ*. Similarly, A lies on *pb QR*, and B lies on *pb PQ* and *pb QR*. By Theorem 1, $A = B$.

To prove the Proposition, if there were three points C, D, E, they would contradict Theorem 3 by lying on two distinct circles, with center A and radius AB, with center B and radius AB.

The argument is not easy, the ideas are

sophisticated, and the axioms on *pb* are extremely ad hoc; they carry us no further. For example, the set of axioms given here is not sufficient to prove that given a pair of points A, B, there is a mid-point i.e., a point C, such that A, C, B are collinear and $AC \equiv CB$. In other words, we cannot prove that *pb* AB is not parallel to the line containing A and B. It would be a sophisticated student, indeed, who felt that this needed proof.

Indeed, it does not! Anyone proving it is only testing his own axiom system. The statement, as applied to intuitive constructs inside my head, is absolutely true. When I write a list of axioms, it is either deducible or not. Is it true in the "real world"? I consider this question meaningless.

The difficult task of giving an adequate axiomatic basis for geometry was performed, inter alia, by Hilbert before 1900. See [3]. Hilbert gave 20 axioms. A list of 13 axioms (the 13th being of a special nature) is given by A. Tarski in [2]. One of Tarski's axioms, 9, translated into more intuitive form, is this: suppose that A, B, C are collinear points, and D a point; then the distance CD is known as soon as the distances AB, BC, AD and BD are known, i.e., given other collinear points A', B', C', and a point D' with $AB \equiv A'B'$, $BC \equiv B'C'$, $AD \equiv A'D'$, $BD \equiv B'D'$; then it follows that $CD \equiv C'D'$.

If these remarks do not show convincingly that geometry cannot be taught deductively at the secondary level, one might consider the following problem: How much would one postulate in order to deduce the class of theorems of which the following is typical? Given collinear points A, B, C, D, with B between A and C, C between B and D, it follows that C is between A and D.

Slightly malapropos, I mention without a moral this anecdote: I was about to present to an analytic geometry class a proof that the graph of $ax+by=c$ is a (straight) line. A student said, "Shouldn't you begin by giving the definition of *line*?" It was an embarrassing moment. Incidentally D'Alembert said in 1750, "It is the disgrace of geometry that it does not supply me with a definition of line." As a final remark, one should not be overwhelmed by Hilbert's 20 axioms. *The Columbia Encyclopedia of Sports* lists 43 axioms for baseball, Hoyle lists 37 for bridge. Hence, geometry is easier than baseball and bridge.

5. Arithmetic and algebra, unlike geometry, are ideally suited for axiomatic presentation because the number of axioms required is small and very easily kept in mind.

We begin by studying a certain duality in arithmetic:

a) (1) The universe of discourse: numbers.
   (2) Facts to be used: i. $0+a=a$, ii. $(-a)+a=0$, iii. $(a+b)+c=a+(b+c)$.
   (3) THEOREM: If $a+b=a+c$, then $b=c$.
   (4) *Proof:* (Step 2 may be justified or not according to taste.) 1) $a+b=a+c$ (given), 2) $(-a)+(a+b)=(-a)+(a+c)$, 3) $[(-a)+a]+b=[(-a+a]+c$ (by iii.), 4) $0+b=0+c$ by ii.), 5) $b=c$ (by i.). Q.E.D.

b) (1) The universe of discourse: nonzero numbers.
   (2) Facts to be used: i. $1 \times a=a$, ii. $\bar{a} \times a=1$, ($\bar{a}$ is being used for convenience to represent $1/a$.) iii. $(a \times b) \times c=a \times (b \times c)$.
   (3) THEOREM: If $a \times b=a \times c$, then $b=c$.
   (4) *Proof:* 1) $a \times b=a \times c$ (given), 2) $\bar{a} \times (a \times b)=\bar{a} \times (a \times c)$, 3) $(\bar{a} \times a) \times b=(\bar{a} \times a) \times c$ (by iii.), 4) $1 \times b=1 \times c$ (by ii.), 5) $b=c$ (by i.). Q.E.D.

The two discussions are obviously identical, and it is uneconomical to do both separately.

Actually, both theories can be presented simultaneously by introducing a neutral notation as follows: Let G stand for the

collection of all numbers (or of all nonzero numbers). Let $*$ stand for $+$ (or $\times$). Let e stand for 0 (or 1). Let $\bar{a}$ stand for $-a$ (or $1/a$).

AXIOMS: i. $e * a = a$, ii. $\bar{a} * a = e$, iii. $(a * b) * c = a * (b * c)$.

THEOREM: If $a * b = a * c$, then $b = c$. *Proof:* $a * b = a * c$, $\bar{a} * (\bar{a} * b) = \bar{a} * (a * c)$, $(\bar{a} * a) * b = (\bar{a} * a) * c$, $e * b = e * c$, $b = c$. Q.E.D.

As soon as desired, one may introduce the axiom $a * b = b * a$.

This game (called group theory) has many amusing aspects. For example, consider the formula

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}. \tag{1}$$

If we translate this into $*$, and thence into $+$, what will we get? The first step yields $(a * \bar{b}) * (c * \bar{d}) = (a * c) * (\overline{b * d})$; the second yields (2) $(a - b) + (c - d) = (a + c) - (b + d)$.

It is rather surprising that the extremely useful and well-known formula (1) should have as its dual the neglected formula (2). Why is (1) so famous and (2) so humble, when they are dual? The answer lies in the fact that the integers are important. Differences of integers are still integers; hence we do not develop a theory of adding and subtracting differences of integers. However, quotients of integers are not always integers; hence we take time to establish rules for working with them. No one cares how to add $(7-3)$ to $(4-9)$; instead, we add 4 to -5. In contrast, everyone learns how to multiply 7/3 by 4/9 and get 28/27.

I remark here that it is probably worthwhile to adopt a neutral symbol for $-$ and $\div$, namely, let $a \sim b$ stand for $a * \bar{b}$. Then, the second step above becomes $(a \sim b) * (c \sim d) = (a * c) \sim (b * d)$.

Since the theories of $+$ and $\times$ are so similar, one expects that there is an isomorphism between them. This turns out to be the logarithm, of course.

6. An axiomatic development such as this is extremely flexible in the following sense: While the argument is proceeding, students of extremely diverse abilities can see and write precisely the same words and symbols. At the same time, they can be thinking thoughts that are worlds apart, but yet, in each case, germane. Hence a teacher can teach an advanced and a regular session simultaneously.

Here are the students' thoughts:

Below average. Thinks everything through twice, once for $+$, and once for $\times$. While following argument, thinks of $*$ as $+$.

Average. Does not have to translate into $+$ and $\times$, but $*$ definitely means one or the other.

Above average. Enjoys playing with $*$ and carrying out a duality procedure like going from (1) to (2) as we just did.

Good. Realizes that "clock-addition" (e.g., $7+8=3$) obeys the axioms. Carries the duality argument into it as well as $+$ and $\times$. Believes the axioms are true.

Bright. Calls G a group and is aware of several particular groups. Wonders how to prove the axioms.

Exceptional. Studies group theory through Lagrange's theorem for enrichment. Realizes the axioms are part of a definition.

All six classes of students must justify every step, learn the nature of proof, and obtain exercise in exact reasoning. I repeat the dictum that this is the only true applied mathematics for the future nonscientist.

7. I offer in the form of a warning and a dialogue a contribution to the problem of selling such a program to the student.

Warning: Do not take the point of view that anything which seems obvious to the student *needs* proof. One turns in disgust from the person who insists, "But how do you know that 1 is not an even number, as well as an odd one?" Yet, one can easily become interested in the problem of deducing this fact from, say Peano's five axioms for the integers.

The situation is this: We all know a large number of facts; for example, if n

is even, $n+1$ is odd; there is no integer between 0 and 1; given any real number, there is an integer larger than it; if $a>0$, then $1/a>0$; $a+b=b+a$; 7 is odd. Let us tackle the problem of classifying our information. Must we write every single fact as an isolated entity? No, clearly some facts follow from others. For example, from "7 is odd" and "if n is odd, $n+2$ is odd," it follows that 9 is odd.

One of the great achievements of the 19th century was the discovery of a very small number of facts from which all the others could be deduced.

Consider this dialogue. *Teacher:* "THEOREM: If $a>0$, then $1/a>0$." *Pupil:* "But that's obvious and doesn't need proof." *Teacher:* (Carefully refrains from saying, "How do you know it's true?") "Yes, but remember our program. We are trying to write down everything we know, either as an axiom or a theorem. Now I happen to know that the result in question can be deduced from the list of facts that we already have. It would be most inelegant to write it as an axiom and increase our list of unproved facts—unnecessarily."

On occasion, of course, some deducible result has a deduction which is so difficult that we abandon our efforts to make our axiom list minimal and simply postulate the result. For example [4], p. 546, last line but one. As Bertrand Russell has pointed out, this has exactly the same advantages as theft over honest toil.

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Teacher exchange program opportunities, 1962-63

Once again it is time for teachers who would like the experience of teaching in a national or an American sponsored school in another country to begin planning for a year abroad. This unusual opportunity is offered to elementary, secondary, and junior college teachers in all subject fields, through the teacher exchange program administered by the Office of Education, in co-operation with the Department of State as authorized by Public Law 584, 79th Congress, the Fulbright Act, and Public Law 402, 80th Congress, the Smith-Mundt Act.

During the 1962–63 school year, approximately four hundred grants will be available in forty countries. The exchange teaching assignments are of two types: those involving an interchange of teachers and those involving a one-way assignment for an American teacher in a school abroad. Summer seminar grants for teachers of French, German, Spanish, Latin, and history will also be available in France, Germany, Colombia, and Italy, and in Brazil and India, respectively.

The basic qualifications for an exchange teacher include the following: United States citizenship, a bachelor's degree, three years of successful teaching experience, good physical health, moral character, emotional stability, maturity, and adaptability. Other considerations being equal, veterans and persons under fifty years of age are given preference.

Being a well-qualified teacher is not enough. A candidate should have a real interest in the culture, history, educational system, and people of the country for which application is made. The teacher should also have the ability and willingness to interpret the United States—its people, customs, and culture—to people abroad.

Applications and complete information regarding teaching and summer seminar opportunities for the 1962–63 academic year can be obtained after August 1 from the Teacher Exchange Section, Office of Education, Department of Health, Education, and Welfare, Washington 25, D.C. Applications will be received until October 15, 1961.

Could we teach limits?

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What is the place of the limit concept in secondary-school mathematics instruction?

IN THE CONCERN for high school mathematics curricula, there are two questions that need separate and careful consideration. First, we must ask whether or not pupils *could* learn a given subject matter, and second, we must ask whether or not they *should* learn the given subject matter. The first question is psychological, and the second is philosophical. The first question is one of prime concern. If the answer to the first is no, there is little need in considering the second.

These questions are being asked continually by responsible people, whether consciously or unconsciously, in their efforts to devise a desirable curriculum.

Maturity has been found to be important in determining that point in an individual's life where certain tasks can be profitably undertaken. "Maturation is the process whereby behavior is modified as a result of growth and development of physical structure: it is illustrated by the fact that until a child is ready to walk or talk, i.e., has the neuromuscular development for making such a performance, no amount of training can bring about these particular changes in performance. Similarly, many higher thought processes are not available to immature humans, i.e., utilization of symbolic notation or algebraic operations."[1]

We often mistake pupils' inability to perform a given task at a given age level as a lack of maturity, as defined above, when in reality it may be that the pupils have not had the necessary background of experience to cope with the given task. In such instances, it is our responsibility to consider and examine the possibility that this is the case.

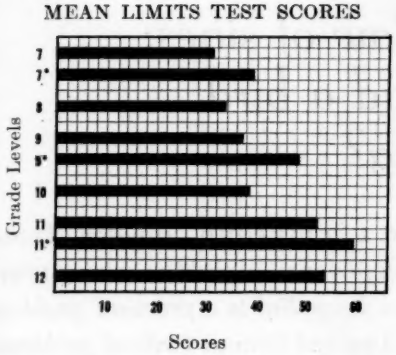
The topic of present concern is the question of whether junior-senior high school students could profit from a discussion of limit. The subject has been traditionally regarded as one which is not sufficiently understood by the pupil to make it worthwhile at the high school level.[2] Yet one must ask, is this due to lack of maturity or lack of experience on the part of the pupil? Could the subject be taught, i.e., is the pupil capable of understanding if given the necessary background of experience.

Acquiring a concept of limit is important to the pupil. A defensible reason for not introducing this concept early in a pupil's mathematical life is that it cannot be sufficiently understood by him to make teaching it worthwhile at the high school level. Is this the case?

An investigation of 578 pupils from grades seven through twelve was made in an effort to determine whether or not high school pupils can profit from experience with the limit concept in mathematics and to determine the relative importance of maturity, as compared to experience, in acquiring a concept of limit. The investigation was limited to pupils who were currently enrolled in mathematics classes.

[1] Howard L. Kingsley, *The Nature and Conditions of Learning*, revised by Ralph Gary 2d ed. (Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1957), p. 12.

[2] National Education Association, *Final Report of the National Committee of Fifteen on Geometry Syllabus* (Chicago: University of Chicago Press, 1912), p. 9.



* Experience Classes.

The Limits Test, an instrument designed through co-operative effort for use in the investigation, was used as an indication of the extent to which an individual had conceptualized limits.

Some of the classes at grade levels seven, nine, and eleven were given three hours of special instruction on limits and were classified as experience classes. The other classes, which received no special instruction, were classified as nonexperience classes. All classes were given the Limits Test. Data on chronological age, mental age, and grade-point average in mathematics classes were obtained for every subject.

Mental age was found to be an acceptable basis for equating groups. At each of the given grade levels—seven, nine, and eleven—the experience and nonexperience classes were equated. The significance of the differences between experience and nonexperience groups in mean Limits Test scores was determined. Significant differences were interpreted to be a consequence of the special instruction.

At each grade level where experience was given, gains made by those pupils who had been given special experience, in the extent to which they had conceptualized the limit, were significant at the .01 level (see graph). This indicates that these junior-senior high school pupils can, indeed, acquire a concept of limit.

While at each grade level where experience was given the difference between the nonexperience and the experience groups was significant at the .01 level, there was no significant difference between grades seven and eight, nine and ten, or eleven and twelve. In each case, it was found that three class hours of experience did produce a difference in the extent to which pupils had conceptualized the limit, whereas a year of added maturity did not. This pattern of differences suggests that, at this age level, experience is the important factor in determining the extent to which an individual has conceptualized the limit in mathematics.

An additional inference drawn from other data is that, in any comparison between grade levels in predicting success in conceptualizing the limit, a matter far more important than consideration of pupils' comparative chronological ages is their comparative backgrounds of experience. In fact, examination of all the factors and variables considered in this investigation indicates that the nature of a pupil's background of experience is far more important in determining his success with problems associated with the limit than any other single factor.

If the subject can be taught profitably, and the evidence here indicates that it can be, because of its importance it is reasonable to suggest that it be introduced in conjunction with those topics now taught in high school which bear on this concept. "Some would eliminate everything which cannot be mastered by the close of a given school period. This attitude implies that a pupil's growth along a particular line does not extend beyond the end of that period."[3] Most of the teaching in mathematics uses the approach of developing ideas along simple, then more and more abstract lines until a working concept is acquired. Should this be the case with the concept of limit?

[3] J. H. Minnick, *Teaching Mathematics in the Secondary Schools* (New York: Prentice-Hall, Inc., 1939), p. 250.

An application of the law of sines: How far must you lead a bird to shoot it on the wing?

EDWARD L. CLIFFORD, *Houston, Texas.*

Important experience in problem-solving where mathematics is applied to a practical problem, an experience not gained through textbook problems, is provided by problems of the type suggested here.

I HAVE HUNTED birds fairly successfully for years without *really* knowing the answer to this question of how much the lead should be. However, this slight lack of knowledge did not restrict me from exercising my "right" as a hunter—that is, the right of taking a few liberties with the truth.

When I was asked by my fellow hunters, "How far did you lead it?" specific answers came easily, since the question was usually asked *after* a particularly beautiful shot. The funny feelings that came over me when I quoted these "authoritative" answers finally got the best of me, so I decided I would try to find out how far I actually lead birds, that is, when I hit them.

While deriving the formula for lead, I could not help remembering my teaching days and how I sometimes groped to find satisfactory application problems for the practical-minded students. This problem impressed me as one that would have "reached" a great many students who, when trying to solve it, would have learned, or acquired experience with, several of the fundamental techniques so valuable in solving all types of practical-application problems. I thought that you might agree that the problem would have the same value to your students today, so I decided to present it.

The entire solution to this simple problem is presented not because it is unique or clever nor to save you the time and effort required to solve it yourself, but to demonstrate the fundamental techniques I have referred to as so important in practical problem-solving. The solution as set forth in the article is also useful if you decide to assign this article to your students as outside reading. Because of this possibility, I have included a few interesting, but unsolved, exercises.

THE PROBLEM

It is obvious, at least to hunters, that there are several factors determining the lead one should apply to a bird in flight. To arrive at a meaningful answer, we have to account for all of these factors. We do this by making certain assumptions, and we make them with care. Of course, when we make these assumptions, we are really trying to understand or form the problem—for without them, we have no problem.

We know intuitively that the lead depends on how far the bird is from us when we shoot at it, his flight path, his flight angle, his size, and, of course, his velocity. Also, obviously involved are the variables of the weapon and shell—shot path, shot velocity, and shot-pattern size.

Analyzing these variables one by one, we can see that all but two can be easily

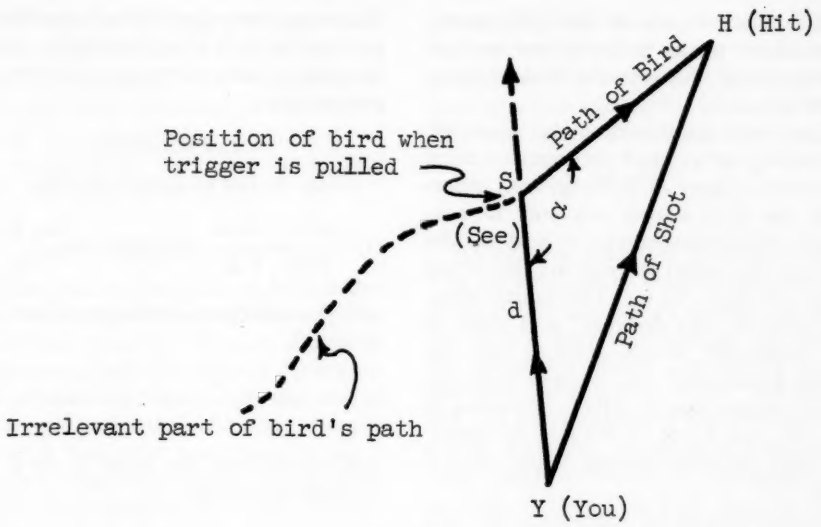


Figure 1

expressed. These two—the path of the bird and the path of the shot—deserve more attention. Knowing that birds may fly in an infinite number of paths, we are inclined at this point to forget the whole thing! A moment's thought, however, coupled with a little hunting experience, tells us that if we are hunting with a gun (and most of us do), the bird does not fly very far between the time we pull the trigger and the time the shot hits him. This helps, for if his flight path is short enough—say a few feet—we can closely approximate it with a straight line. And too, if the bird is within gun range, the flight path of the shot can also be represented with a straight line without seriously affecting the accuracy of our answer.

The problem takes shape. With the help of a more concise statement of our assumptions and a sketch, it should be in true focus. (See Figure 1.) Let us say then

1. The bird is a very small object at a distance of d feet ($d \geqslant$ gun range) from the hunter, or, more precisely, from the shell in the hunter's gun. (The bird's size obviously helps determine lead, but, since it complicates the problem without adding much to the answer, we will neglect it throughout the solution and assume we have to hit him dead center.)
2. It is flying at a velocity V_B in a straight line at some angle α measured from the hunter's line of sight YS to the path of the bird SH.
3. The shot from the gun travels in a straight line at velocity V_S, and (until we get to shotgun leads later on in the article) we will also say the shot charge or pattern is very small.

If we let t_B represent the time of flight of the bird after we pull the trigger (his flight time from S to H), and t_S the time of flight of the shot (from Y to H), we can express the distances SH and YH respectively with the products $V_B t_B$ and $V_S t_S$. Another sketch, Figure 2 on the following page, shows this.

We are feeling up to our task now with all these "knowns." With a triangle of three known sides and one known angle, we should be able to "whip out" quickly the answer to our original question. We would, too, if it were not for one small fact—we have forgotten to define lead and

that is what we are seeking! Of course, since we are trying to decide *how much* of it is necessary, it is essential that we know *what "it" is*.

Even more caution than that exercised in making our original assumptions must be exercised here in defining lead; otherwise, our final answer will only be academic. Most hunters refer to lead as "the distance you point the gun in front of the bird in order to hit it." This sounds clear, but what is "the distance" and where is "in front of?" Is the distance SH, the bird's flight path? No, because when the bird is going away ($\alpha = 180°$) or coming in ($\alpha = 0°$), we know no lead is necessary, whereas SH is several feet. Then, what is lead?

To the best of my knowledge, it is the "apparent travel distance" of the bird, that is, the perpendicular distance between the position of the bird (S) and our line of sight (YH) down the gun barrel when we pull the trigger. Said another way, it is one of the altitudes in ΔYSH. We hereafter use this definition as the correct one and denote it by L, which is shown on our final sketch. Point A and angle β have also been added for reference purposes in Figure 3.

The analysis is over, so we can quickly derive our expression for lead, provided we are familiar with a few operations and relationships from elementary algebra and trigonometry.

The solution

Using the law of sines in ΔYSH,

$$(1) \qquad \frac{\sin\alpha}{V_S t_S} = \frac{\sin\beta}{V_B t_B} \quad \text{or} \quad \sin\beta = \frac{V_B t_B \sin\alpha}{V_S t_S},$$

and the definition of the sine of an angle in ΔYSA,

$$(2) \qquad \sin\beta = \frac{L}{d} \quad \text{or} \quad L = d\sin\beta,$$

and substituting the value of $\sin\beta$ from (1) into (2), we get

$$(3) \qquad L = \frac{dV_B t_B \sin\alpha}{V_S t_S}.$$

Naturally, the flight time of the shot equals the flight time of the bird—that is,

$$t_S \equiv t_B,$$

so our formula for lead becomes

$$(4) \qquad L = \frac{dV_B \sin\alpha}{V_S}.$$

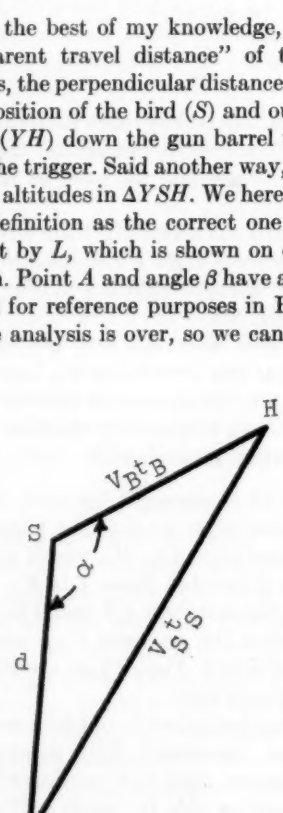


Figure 2

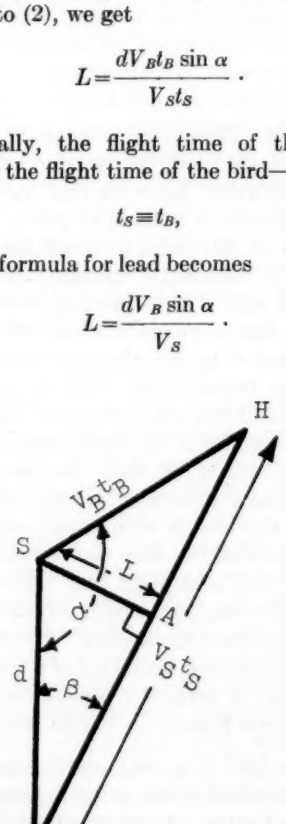


Figure 3

This lead* is the lead a hunter would have to apply if the shot pattern was very small—for example, if the "shot pattern" was a rifle bullet. However, in order to hit moving birds, hunters have found it necessary to use larger shot patterns to help "cover up" their errors in lead.

Lead applied to shotguns

Obviously, the true lead, or dead-center lead, expressed in formula (4) is always the proper lead, even for shotguns. But with shotgun areal patterns, there is a permissible lead error. If we calculate the size of the shot pattern and combine the relevant part of it with true lead, we will get a formula for the lead necessary to hit birds, that is, a functional lead. To do this, we need to make a few other assumptions that can be found approximately correct by field-testing them.

Let us say that the pellet pattern emerging from the shotgun is circular, expanding linearly as it travels outward. (Actually, the path it cuts out approximates a right circular cone, but the pellet pattern at any instant is not circular because the pellets are "strung out" in the direction they are traveling.) Assume also that at every distance within gun range, the pellets are evenly distributed and dense enough to hit the bird anywhere within the pattern. Then, the shot pattern's radius at any distance $V_S t_S$ along the shot path will be $kV_S t_S$, where k is a constant determined primarily by the choke of the gun and somewhat by the shell type. Since $V_S t_S \cong d$ (not shown by the misleading, exaggerated scale in our sketches), the shot pattern's radius is approximately kd. Adding or subtracting this permissible error to our dead-center lead, depending on whether we are underleading or overleading the bird, we get the limits of functional lead (L_f) for shotguns:

$$L_f = \frac{dV_B \sin \alpha}{V_S} \pm kd.$$

We can see that true lead gets larger with distance, but so does the pattern size. Is it possible that actual values of k—gun and shell constants—eliminate the necessity of lead altogether? The answer to this question and others is left to the reader and can be obtained by working the following exercises. (Exercise 6 requires a little practical research for the gun hunters; but, if they take their hunting seriously, they would probably want to do it anyway.)

Exercises

1. Do you have a better definition for lead? If so, derive a formula expressing it.
2. If we assume the bird is in range and a shotgun is being used, when, if ever, is any lead necessary? (Eliminate the flight angle by letting $\alpha = 90°$, the angle which maximizes the lead.)
3. Now, using your answer from Exercise 2, determine how fast a bird has to be flying before you have to lead him at all if you are shooting a 1200-ft/sec shell, and your gun patterns this shell over a circle of diameter 4 ft. at a distance of 30 yd.
4. How many flight angles (α) require the same absolute lead?
5. High-powered 12-gauge shells have a velocity of about 1000 ft/sec. How far should you lead (dead-center lead) a bird with this shell if it is 35 yd. away, flying at a 120° angle at a velocity of 50 mph?
6. Look up the velocity of the shell you normally shoot. Determine k for this shell in your gun. Using this V_S and k, see if you can find a value for lead that

* One must remember that there is a time-delay interval between the instant your brain tells your finger to pull the trigger and the instant your finger actually pulls the trigger. It also takes time for the firing pin to reach the shell after your finger pulls the trigger. Because of these delays, if a hunter stops his gun to shoot, a little extra lead will have to be added to the lead expressed in formula (4); otherwise, he will shoot behind the bird. However, if he uses the recommended follow-through technique and shoots while he is swinging with the bird, no extra lead is necessary.

will apply to nearly any bird you shoot on the wing. (Let d and V_B vary within reasonable limits.)

7. Is our lead formula applicable to other similar problems—for example, to the lead a forward passer applies to his receiver going down and out?

Concluding remarks

This particular problem has several properties that should make it interesting to many students. First of all, it is about something in *their* everyday lives. Hunting and the leads to apply to birds are common topics around the school.

For those students who hunt, the solution to this problem seemingly answers an important question. Since trigonometry age is the usual age at which students begin to hunt, and since this problem is basically a trigonometry problem applying the law of sines, it would probably create the most interest if presented to a trigonometry group right after they study the law of sines. This time is particularly appropriate since it usually coincides with the hunting season.

The most important property of this problem is its simplicity—it can be easily solved and completely understood. Unfortunately, but necessarily, most textbook application problems deal with scientific topics unfamiliar to elementary mathematics students. As a result, these students do not *fully* understand the answers to such problems, even when they solve them. Manipulative skills are developed, but the confident feeling that one experiences when he completely solves and understands a problem is not provided for. And, too, since he is not familiar with the scientific topic, the "application" problem may not convince him of the usefulness of mathematics. Many students, without this feeling of confidence and assurance of the usefulness of mathematics, cannot develop the interest and drive it takes to become an efficient problem-solver.

I hope that by presenting my solution I have illustrated several important techniques which students will encounter when trying to answer the same question, that is, if you present it in word form and refuse to give any further assistance. Notice, for example, that I spent a lot more time and effort trying to understand the problem than I did in actually solving it. This is often the case and should not be avoided, since it is something we *have* to do whether we want to or not. When students realize that understanding the problem is often not *part* of the problem, but *the problem itself*, they will devote more time (as they should) to understanding.

If you do give this problem in vaguely worded form, the way problems are usually encountered outside the classroom, students in trying to understand it will have to form their own assumptions and even their own definitions—realistic ones, too, if they want realistic answers. They will have to check their own solutions, which is also a desirable practice.

When trying to set up realistic assumptions and definitions, they will "discover" sketches, if they have not discovered them already. They will find that sketches, like equations, relate the variables, simplify the problem, and are, in general, wonderful problem-solving aids.

Perhaps problems of this type would have to be assigned as outside elementary research to be reported on in theme form. If they were, another concept often overlooked in mathematics courses would be gotten across to the students. They would learn that writing a clear description helps clarify a problem to the writer as much as it does to the reader. By writing their answers in theme form, the students would also get valuable training in exposition.

I realize that this problem is a special problem, but several other problems with similar properties probably exist. If a sufficient number could be recognized, solved, and catalogued, I think they would serve as invaluable teaching aids in a field that could stand them, especially if the future turns out to be as scientific tomorrow as it looks to us today.

Introducing proof with a finite system

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Students should be taught to be curious,
so they will ask the question,
"Is this all there is to the matter?"

AN INTRODUCTION to high school geometry should include not only an explanation of the meaning of proof in mathematics but also examples of such proof. The examples chosen should be well within the experience of 10th-grade students. This suggests something from Algebra I, with which the students are most familiar.

Proportions provide rather easy theorems as examples of proof and also an opportunity to illustrate the modern mathematical point of view. The meaning of proof to be presented is that certain statements are selected and arranged in a sequence of statements, that is, linked together in a chain of reasoning, with the last statement the desired theorem.* This meaning of proof can be illustrated by the following theorem, known to the students as "alternating the means of a proportion":

1. $\frac{n_1}{n_2}=\frac{n_3}{n_4}$	1. Assumed
2. $\frac{n_1n_2}{n_3n_2}=\frac{n_3n_2}{n_3n_4}$	2. Multiply by n_2, and divide by n_3
3. $\frac{n_1}{n_3}=\frac{n_2}{n_4}$	3. Dividing numerator and denominator by the same number

* For a more complete statement of the meaning of proof, see: Myron F. Rosskopf and Robert M. Exner, "Modern emphases in the teaching of geometry," THE MATHEMATICS TEACHER, L (April 1957), 272–279. The pattern of proof used is the "Statement-Analysis" form presented in this article.

This proof, while not completely rigorous, should be adequate for the purpose stated. There are two other theorems that may also be used as examples, namely, "alternating the extremes of a proportion" and "inverting a proportion."

These three theorems are enough to show students what theorems look like. However, neither the teacher nor the students should be satisfied with just this much information. Two basic questions remain unanswered:

1. Are these all the possible theorems?
2. How are these theorems related to each other?

Securing answers to these questions and similar questions reflects the modern mathematical point of view.

The answer to the first question is that the three familiar theorems are not all the theorems that can be proved. For example, the proportion

$$\frac{1}{2}=\frac{4}{8}$$

may be changed into the following proportion:

$$\frac{2}{8}=\frac{1}{4},$$

and this is not one of the familiar theorems. Since each numeral has apparently been moved clockwise to replace the next numeral, this theorem may be called "a

proportion obtained by rotating a proportion." Further investigation reveals that this theorem is not the only new theorem obtainable by "rotation." If the preceding theorem is symbolized by "R_1," that is, rotation of one place clockwise, then the remaining theorems are "R_2," rotation of 2 places clockwise, and "R_3," rotation of 3 places clockwise.

Thus, there are seven theorems about a proportion with the restriction that the 4 numbers remain the same. These seven theorems are: R_1, R_2, R_3, alternating the means (M), alternating the extremes (E), inversion (I), and reversing the sides of a proportion (S).* This completes the answer to the first question.

Now what are the relations between this set of transformations on a proportion, that is, $\{R_1, R_2, R_3, M, E, I, S\}$? First, let us consider only the changes involving rotation. To begin with, there is no reason why the rotation had to be clockwise, since it could just as well have been counterclockwise. Such a rotation may conveniently be symbolized as "R_{-1}," "R_{-2}," etc. Now what happens if the numerals are rotated four places, either R_4 or R_{-4}? The answer is, nothing, for the proportion remains the same. This suggests the symbol "R_0" with

$$R_0 = R_4 = R_{-4}.$$

What happens if the numerals are rotated 5 places or more? R_5 produces the same proportion as R_1, R_6 produces R_2, and so on.

Thus, there are just 4 rotations in this system:

$$R_1 = R_{-3}$$
$$R_2 = R_{-2}$$
$$R_3 = R_{-1}$$
$$R_4 = R_{-4} = R_0.$$

More important yet is the fact that these four elements (rotations) together with (followed by) the undefined operation, form a finite cyclic group. This is demonstrated as follows:

1. Each rotation followed by a second rotation is equal to a rotation. For example, $R_1 \cdot R_2 = R_3$. (The symbol "$\cdot$" is read "followed by.") Thus, the system is closed.
2. There is a unique identity rotation, R_0. Thus, $R_i \cdot R_0 = R_i$, for all i.
3. Each rotation has a unique inverse rotation. For example, $R_1 \cdot R_{-1} = R_0$.
4. The operation "followed by" is associative. This means, for example, $R_1(R_2 \cdot R_3) = (R_1 \cdot R_2)R_3$.
5. In addition, the system is a commutative group, since $R_i \cdot R_j = R_j \cdot R_i$ for all i, j.

Next, what about the set of transformations: $\{R_0, R_1, R_2, R_3, M, E, I, S\}$? This set, too, forms a group under the operation "followed by," but the group is noncommutative. The reader may be interested in verifying, for example, that $M \cdot R_1 \neq R_1 \cdot M$. It is interesting to note that any pair of theorems which is equal to R_2, or includes R_2, is commutative. Thus, $M \cdot R_2 = R_2 \cdot M$ and $M \cdot E = R_2 = E \cdot M$.

These equations provide interesting exercises for students as well as excellent practice in handling symbols. This analysis may also be extended to false proportions or inequalities. Briefly, the rotation of a false proportion is again a false proportion, and so on to produce another finite cyclic group.

What is the purpose of going beyond the familiar theorems on proportions in this introduction to geometry? There are two purposes which correspond to the two questions asked in the beginning of this article: (1) Students should be taught to be curious, so that they will think to ask the question, "Is this all there is to the matter?" (2) Students should be taught how a mathematician looks at things in relation to each other. This means that students should at least be introduced to the concept of a group.

* The symbol "S" is used since this is the symmetric property of an equation.

• EXPERIMENTAL PROGRAMS

Edited by Eugene D. Nichols, Florida State University, Tallahassee, Florida

High school geometry via ruler-and-protractor axioms—report on a classroom trial

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There is considerable talk these days about curriculum "reform" in secondary-school mathematics. From our past experience, however, we can infer that such reforms will be very difficult to implement; witness, for example, the stability of solid geometry as a semester course in the twelfth grade despite repeated recommendations over a number of years for change to something more fruitful and useful.* It seems obvious that it will take some years to effect any widespread revision of mathematics instruction and have this revision reflected in textbooks and "official" courses of study.

In this situation, it would seem up to the individual mathematics teacher to examine the courses which he now teaches to see what he can do about experimenting with the use of some of the suggested new materials within his courses as they are now organized. Furthermore, such experimentation is altogether a necessary thing in order to develop a backlog of materials that have been tried in the classroom and that have been tested for suitability (or unsuitability) for secondary-school students. Without a backlog of such experimentation there is some danger that the much touted curriculum "reform" may merely consist of replacing the unsuitable parts of today's curriculum with equally barren material.

Thanks to an appointment to the 1957–

* The unpopularity with students and the possible unfruitfulness of a semester course in formal solid geometry was suggested by a 1920 Commission of the College Entrance Examination Board. It suggested replacement of most of the proofs of solid geometry with work on mensuration, space perception, and appeal to imagination and suggested the possibility of combining this work with the plane geometry course. The CEEB provided an examination for such a unified course. W. R. Longley, in the Fifth Yearbook of the NCTM (1930), notes that this examination was almost never elected and comments on the need for " . . . rearrangement of the mathematics curriculum which will offer to the student in the last year of preparatory school a course which is more attractive than the present ones and which will better prepare them for continuing the study of mathematics in college." The final report of the Joint Commission of the MAA and NCTM, published as the Fifteenth Yearbook of the NCTM (1940), recommends for the twelfth year " . . . a unified course from the fields of advanced algebra, solid geometry, analytic geometry, trigonometry, and differential calculus," with the "core," or main emphasis, being advanced algebra. (Note that the recommendation did not envision abandoning solid geometry completely, but had the modest aim of abandoning it as a full-semester course and improving the twelfth-grade offering.) Yet, a 1953 report from the U.S. Department of Health, Education, and Welfare (Bulletin No. 5, 1953) reveals that where a twelfth-grade course was offered at all, it was almost certain to consist of separate one-semester courses in trigonometry and solid geometry. The July, 1956, *Bulletin of the California State Department of Education* shows at that time 308 California high schools offering trigonometry, 238 offering solid geometry (compared with 447 schools offering plane geometry), and a very few schools offering other kinds of twelfth-grade mathematics courses. This was probably typical of the rest of the country.

58 University of Chicago Academic Year Institute for High School Mathematics Teachers, the author had the opportunity to carry out an examination of the plane geometry course as he has taught it for several years. As a result of this examination and the weaknesses it revealed in the traditional course, another approach was formulated and teaching materials written. These materials were used experimentally during the 1958–59 school year. This paper has been written in the belief that a description of this investigation might be of some interest to teachers of mathematics. The approach adopted is being currently advocated as a replacement for the traditional "Euclidean" course. The SMSG geometry materials, for example, use a variation of the approach described here.

In making an investigation of this type, the teacher of geometry is confronted with a number of different and often contradictory statements of the purposes and objectives of the plane geometry course. A currently popular view, for example, is that one of the primary aims ought to be the development of "reasoning in life situations." But some people have seriously questioned whether it is really possible in a one-year course to do an adequate job of teaching "reasoning in life situations" without neglecting the geometric content or obscuring the use of deductive methods in mathematics. These people suggest that geometry is not the only place where students develop sound reasoning in life situations, nor necessarily the best place. Some have proposed to skirt this issue by making the "practical" uses of geometry the main emphasis and greatly de-emphasizing deductive reasoning and "proof." These people would rely primarily on intuitive understanding. Others have suggested cutting the traditional geometry down to the bare minimum needed for the introduction of analytic geometry, but others have strenuously objected to such an emasculation of traditional demonstrative geometry.

In considering these contradictory opinions, it is clearly necessary to make some choices. The choices made by the author of this paper and his frame of reference are summarized as follows:

1. The two main emphases of the geometry course should be: (*a*) the geometric content, i.e., the *facts* and *relationships* of geometry; (*b*) the use of deductive methods as *applied in mathematics*. "Reasoning in life situations" may occasionally be brought up, but should not become the aim of the course.
2. There are far too many values in the traditional synthetic course, especially with respect to the aims mentioned above, to revise it in the fashions advocated either by those who would eliminate proofs in favor of intuitive understandings or those who would virtually eliminate synthetic geometry in favor of analytic geometry.
3. The student in his study of geometry should have the opportunity to maintain and extend his previously learned mathematical skills from arithmetic and algebra. He should gain appreciation of the fact that arithmetic, algebra, geometry, and trigonometry are not separate and distinct, but that they are related in many fundamental respects. It should be noted here that the traditional course in geometry generally fails to do this.

In spite of the writer's bias for retention of some kind of "demonstrative" geometry with the usual structure of axioms and theorems, careful investigation of the traditional course forced him to decide that the traditional course is seriously deficient in several respects. Some of these faults are related to the organization of the course, including the vast number of theorems required by most courses of study. These theorems seem mainly to make the game of proof tedious and repetitious, to obscure the role of logic in the development of the course, and to en-

courage memorization of theorems and proofs. Other and probably more basic faults are related to the almost fanatic adherence in the traditional course to the so-called Euclidean tradition. This tradition handicaps the student with a treatment of the foundations of geometry that is quite inadequate from mathematical and pedagogical standpoints. In support of this contention, consider the following points:

1. George D. Birkhoff and Ralph Beatley wrote in 1930: "The traditional approach to demonstrative geometry involves careful study of certain theorems which the beginner is eager to accept without proof and which he might properly be led to take for granted as assumptions or postulates. Such an approach obscures at the very outset the meaning of 'proof' and 'demonstration.' The employment of superposition in some of these theorems is even more demoralizing. This method of proof is so out of harmony with the larger aim of geometry instruction that despite its validity its use is commonly restricted to those few cases for which no better method can be found. [1]†
2. The "Euclidean" presentation ignores, or does not admit as valid, consideration of "measurement" of line segments or angles. The emphasis is put on unmarked instruments, and the pretension is made that measurement does not exist—at least until fairly late in the course. Such an approach flies in the face of the student's intuition and experience.
3. Until fairly late in the usual course no use is made of the very powerful arithmetic and algebraic techniques in which the student at this level is presumably well trained. Yet most of the "assumptions" listed by the standard text for operation with the "things" of geometry, i.e., unmeasurable line segments and angles, are really disguised versions of the rules for operations within the real number system! Thus geometry is made to seem like a completely separate discipline with no connection whatever with the mathematics learned earlier. The student's other mathematical skills are allowed to deteriorate and be forgotten so that when algebraic methods are finally admitted and used late in the course, it is generally necessary to take time to review and reteach much of the algebra that has been previously taught. Incidentally, in this review it is usually a revelation to the students to learn that algebra is also a mathematical system with axioms and theorems!
4. We cannot even defend the present treatment with argument that it is the Euclidean treatment that has sufficed for thousands of years and is an example of a perfect, logical system. In the first place, the conventional treatment is not "Euclidean," as even a casual comparison of it with Euclid's *Elements* makes obvious. In the second place, Euclid's treatment has been shown by later investigators, e.g., Hilbert, to have serious errors and omissions in its logical structure. The present textbooks dispense with most of Euclid but retain the errors and omissions.
5. Present textbooks seem unable to give any sensible treatment to that old bugaboo of "the incommensurable case." Since Euclid did not possess the tools of arithmetic and algebra (which all our geometry students now have), he was able to handle this only by an exceedingly shrewd definition of proportion which is considered too difficult for secondary-school pupils. The practice of present texts is to prove theorems for the commensurable case and "assume" that they will then hold for the incommensurable case. But if it is mentioned at all, it can hardly be considered as exceptional because the incommensurable case is relatively much

† Numbers in brackets refer to references given at the end of the article.

more common than the commensurable. The treatment of the foundations that will be outlined later in this paper suggests a satisfactory way out of this dilemma.

If we accept the validity of the foregoing arguments against the conventional treatment, what shall we use to replace it? Felix Klein, in 1900, gave us a clue in his *Elementary Mathematics from an Advanced Standpoint.* [3] After a discussion of a treatment of plane geometry via rigid motions which is logically complete, but probably too sophisticated for the secondary-school learner, he suggests another approach.

1. We begin, just as before, with the introduction of *points* and *lines* and theorems concerning their *connection, order,* and *continuity.* (These theorems include such things as "There is one and only one straight line through two given points.")
2. By the side of these—and this is new here—we assume the new fundamental notions, on one hand, of *the distance between two points* (segment), and, on the other hand, *the angle between two lines;* and we set up axioms concerning them which state, in effect, that segments and angles can be measured by numbers in the customary manner.
3. Here the first congruence theorem appears as the following characteristic axiom, which really replaces the axioms of the group of motions: *If two triangles have two sides and the included angle respectively equal, they are congruent,* i.e., they are equal in all their parts.
4. For the rest, the procedure is opposite to that in our first development (via rigid motions). It is customary first to prove the Pythagorean Theorem, and then to introduce the trigonometric functions cosine and sine. From this beginning, the same analytic apparatus is finally derived that we have stated.
5. In this process it becomes necessary to set up another axiom which is very important concerning the theory of parallels . . . we postulate that *there is only one parallel through a given point to a given line.*

An approach that is somewhat similar to the one just described has been developed in some detail by George Birkhoff and Ralph Beatley. This is the exposition that was most influential in shaping the approach described here. It was first presented in a form intended for teachers of geometry in the Fifth Yearbook of the National Council of Teachers of Mathematics in 1930. [1] A course of study based on this exposition was taught as an experimental course through the thirties and was published as a high school text in 1940. [2] Unfortunately, it did not catch on and went out of print. This approach can be best described in the author's own words, so we will quote briefly from their Fifth Yearbook article.

> . . . we take for granted at the outset the notion of number and assume that the student is capable of making simple computations by means of number. Also we admit the self-evident fact of linear and angular mensuration and scale drawing; that is to say, we accept the simple facts familiar to any boy or girl who knows how to use ruler and protractor.

Birkhoff and Beatley then list five "Fundamental Principles" which in effect postulate the validity of linear and angular measurement and side-angle-side similarity of triangles. The use of a similarity axiom (SAS) to replace the congruence axiom and the parallel postulate is the main difference from the Klein approach just described. The authors then use their five fundamental principles to prove six "Basic Theorems" that include side-side-side and angle-side-angle similarity (congruence is considered as a special case), the sum of the angles of a triangle is 180 degrees and the Pythagorean Theorem. From these theorems and their basic assumptions, they develop the remainder of geometry. They defend their system in these words:

> We have suggested a modification of Euclid in accord with psychological and mathematical ideas which, though commonplaces for us today, were not available to Euclid. This modification would at once simplify the subject and give it greater significance. . . .
>
> In the first place, it is severely logical. In this respect it offers as satisfactory training as that of Euclid, and is much more direct. It lends itself admirably to explicit consideration of the place of undefined terms, definitions, and assumptions in any chain of logical reasoning and leads to the development of those same habits, attitudes, and appreciations which all teachers of geometry claim for their subject. In the second place, it takes advantage of the knowledge of number and of linear and angular mensuration which the student possesses, and so does away with the feeling of artificiality which is inevitable when seemingly self-evident propositions are "proved." In the third place, it leads very

naturally to the elementary facts of analytic geometry and makes it apparent in this way that geometry is really a self-consistent discipline whether or not such things as points, lines, and planes really exist.

The writer of this paper is convinced that the approach outlined by Birkhoff and Beatley does have all the advantages claimed for it and more as well. It overcomes in elegant fashion virtually all the criticisms listed here for the conventional geometry course. Consequently, the Birkhoff-Beatley point of view was adopted in formulating the writer's geometry program. The actual "text" presentation of the writer's program differed, however, in several respects from the published textbook presentation of Birkhoff-Beatley. These differences will now be considered.

The Birkhoff-Beatley text has been criticized by some because the postulates concerning measurement of angles and line segments implicitly require assumption of the axioms and theorems about properties of the real number system. Yet these are not explicitly stated for the student and, furthermore, the student is not, under present methods of instruction, accustomed to thinking of the number system in this fashion. The writer of this paper has attempted to overcome this difficulty by first discussing the characteristics of deductive methods in mathematics generally, and then explicitly treating the real number system axioms and theorems as an example of a mathematical system in which deductive methods are used. A lengthy formal treatment is not given. This part of the course merely lists the undefined terms, relies on the student's intuition and experience in listing the axioms of the system (commutative law, identities, etc.), and points out that from these few axioms and undefined terms the other laws can be deduced as theorems. The most important of these theorems are listed and a few are proved, e.g., $(-a)\ (-b) = ab$. Such an approach does double duty in that it gives the student in the very beginning a familiar example of what constitutes a deductive system, and it explicitly lists the propositions about numbers that will be assumed in the treatment of geometry from the point of view described.

Another difficulty with the Birkhoff-Beatley presentation seemed to be that in starting out with similarity students are forced to do the hardest things, from the beginning student's standpoint, first. Yet this approach via similarity is certainly the most elegant one and, in the long run, the simplest one. The writer attempts to solve this dilemma by first stating the similarity propositions in terms of congruence, then offering a section of work on congruence. Thus the student has an opportunity to become acquainted with geometric proof with the relatively simple examples provided by the congruence concept. The course then returns to these propositions in terms of similarity and points out that congruence is just a special case of this concept. The development from this point on follows that of Birkhoff-Beatley fairly closely. The similarity propositions, along with the measurement postulates, are the keys that unlock the knottiest of geometric problems in a way easily understood by the student.

The foregoing compromises had the effect of making it possible to use a standard text as a source of much of the problem material, although the order of propositions in the writer's experimental course somewhat differs from a standard text. Many of the proofs are also different since they use the concepts we have mentioned. But it was possible to use the standard text along with the introduction described here for much of the work of geometry up to a unit on analytic geometry. This afforded an altogether necessary saving of the teacher's time.

The other major innovation attempted was a substantial unit on analytic geometry and proofs of theorems by analytic means. We note that the recommendations of the Commission on Mathematics in this respect have been subjected to criticism by those who fear that demon-

strative geometry will be emasculated and made ineffective to provide room for the analytic geometry. These people rightly argue that there are too many values in demonstrative geometry to allow this to happen. The approach recommended here, in the opinion of the writer, effectively circumvents such criticisms. The economies achieved by the simplified and more powerful set of foundation axioms and the selection of the more important theorems left ample room for the addition of analytic geometry without damaging the real values of demonstrative geometry. Furthermore, this set of axioms has built into it the essential concepts necessary for setting up a co-ordinate system for analytic geometry. Thus economies of time are also effected in the presentation of the unit on analytic geometry.

The course of study that developed from these deliberations is briefly outlined as follows:

1. Discussion of deductive methods and need for undefined terms, defined terms, axioms, and theorems.
2. Review of the properties of the real number system under the usual operations, putting it in the form of a deductive system such as we have described.
3. Introduction of the axioms for the geometric system. As implied above, these differ considerably from the usual axioms because they assume the validity of "measurement" of line segments and angles in terms of numbers and take as part of the geometric system all the previously listed axioms and theorems of the real number system. Side-angle-side similarity and congruence are assumed.
4. Proof of a selected list of theorems covering the content of the traditional course. About forty theorems are given prominence in the text; others appear as exercises for the student. This part of the course follows the standard text quite closely, except that similarity is treated early, just after congruence. Thus the essential foundations for the entire course are covered in the beginning stages. The axioms and theorems are:

a) (Axiom) Principle of linear measure: (1) To each pair of points there corresponds a distance which can be expressed as a positive number. (2) The points on a line can be numbered so that number differences measure distance.

b) (Axiom) There is one and only one straight line through two given points.

c) (Theorem) Two lines can have at most one point in common.

d) (Axiom) Principle of angle measure: (1) To every angle there corresponds a number modulus 360. (2) To every number there corresponds an angle. (3) All rays from the same point can be numbered so that number differences measure angles.

e) (Axiom) All straight angles are equal; two rays that form a straight angle also form a line; and any line can be considered as a straight angle by marking a vertex point on it.

f) (Axiom) (1) SAS congruence, (2) SAS similarity.

g) (Theorem) (1) ASA congruence, (2) AA similarity.

h) (Theorem) If two sides of a triangle are equal, then the angles opposite these sides are equal.

i) (Theorem) If two angles of a triangle are equal, then the sides opposite these angles are equal.

j) (Theorem) (1) SSS congruence, (2) SSS similarity.

k) (Theorem) All points equidistant from the ends of a line segment, and no others, lie on the perpendicular bisector of the line segment.

l) (Theorem) The sum of the angles of a triangle is 180 degrees.

m) (Theorem) Through a point not on a line there is one and only one perpendicular to the line.

TABLE 1
SHAYCROFT PLANE GEOMETRY TEST—RAW SCORES

	Experimental	A	B	C	D	F
Median	46	45	42	42	46	46
Range	25–59	26–54	30–57	25–55	35–54	35–56

n) (Theorem) The Pythagorean Theorem.

o) (Theorem) Through a point not on a line, one and only one line can be drawn which does not meet the given line.

5. The two-dimensional, co-ordinate system is introduced using the axiom on linear measure and the necessary theorems proved in Section 4. The main concepts of analytic geometry are covered, including distance, slope, and midpoint formulas; criteria for parallelism and perpendicularity; various forms of the equations for lines; intersection of lines; and special and general forms for the equation of a circle. Some theorems from plane geometry are proved by analytic methods, including the usual concurrence theorems for medians, perpendicular bisectors, and altitudes of a triangle.
6. The course ends with a re-examination of the chain of axioms and theorems that make up the system. Certain theorems that were considered too difficult to prove in the beginning stages and were thus temporarily assumed (such as "the parallel postulate," SSS similarity, etc.) are proved as part of this reconsideration of the system.
7. When appropriate throughout the year, concepts are extended to three-dimensional space, primarily by analogy and appeal to intuition.

As to the outcome of the course in an actual teaching situation, no reader will be surprised to learn that the writer felt it quite successful. Certainly there were unexpected difficulties and sticky spots, but these were counterbalanced by other things that went better than he had hoped. One example is the consideration of algebra as a deductive system, another example is the ease with which pupils accepted the axioms and the economies this introduced into the study of similarity. It proved to be a very efficient system—it was not at all difficult to cover the conventional material and still spend six weeks on the co-ordinate geometry and several weeks on other more or less optional topics. If the reader wonders whether the conventional material was really learned or, as is possible, merely sacrificed for the "new" material, the evidence of standardized tests indicates that it was learned.

The Shaycroft Plane Geometry Test was given to all six geometry classes in the school in the last week of the school year. The school is a comprehensive public high school in the San Francisco Bay Area. The "experimental group" was assigned on a purely random basis in the scheduling process, and certain test evidence indicates that it was about the same in composition as the other classes. The test data are given in Table 1.

In such a limited experiment it is pointless to do statistical gymnastics and make claims on the basis of the figures, but there is an indication that the experimental group learned the conventional material at least as well as the other groups, even though they spent considerably less time on it. In addition, local tests indicate that the experimental group gained a good understanding of basic co-ordinate geometry. The experimental students also were probably able to maintain and extend their algebraic skills and understanding. It is the feeling of the writer that with polishing, the system

would become even more efficient, thus leaving even more time for the numerous topics that can be profitably correlated with a geometry course.

This paper has been concerned with one teacher's examination of his plane geometry course and his proposed solution to some of the inadequacies of the conventional treatment of that course. As stated at the beginning, much experimentation such as this is needed before proposed reforms and proposed new materials become fixed in our mathematics curricula.

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Books for Asian Students program

Titles in algebra, geometry, trigonometry, and calculus are needed by Asian students and libraries. You may donate books in good condition published after 1945 to The Asia Foundation's book program. Please do not include mathematics books which contain mainly materials applicable only to American students, such as dollar and cents calculations, taxes, and other American economic details.

Books may be sent by educational materials rate post in packages under 70 pounds in weight. Ordinary cardboard cartons are adequate for packing. For bulk lots of books meeting the program's criteria for date and condition The Asia Foundation will meet the shipping expenses. All shipments, and queries concerning procedures for bulk shipments, should be addressed to:

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The Asia Foundation is a nonprofit, nonpolitical organization established by private American citizens to strengthen Asian educational, cultural, and civic activities with private assistance.

Teaching machine research

System Development Corporation (SDC) of Santa Monica, California has received an $88,283 grant from the United States Office of Education for research on the use of a computer-based teaching machine designed to select and organize educational material for high school students.

Experimental evidence obtained will indicate whether automated teaching devices and materials will facilitate learning in the rote and conceptual aspects of high school mathematics. In addition, evidence will show whether such techniques are effective for students who have not been performing at a level considered proportionate to their basic aptitudes and intelligence quotients. These underachievers will be compared with normal and overachieving students to see if the automated instructional materials have benefited them more.

In automated teaching, as with any other educational aid or technique, the quality of instructional material is the most important consideration. SDC researchers believe that there is a great need to determine the effectiveness of different categories of self-instructional materials for different groups of students and to study the relationships among these student-material variables in the actual classroom environment.

• HISTORICALLY SPEAKING,—

Edited by Howard Eves, University of Maine, Orono, Maine

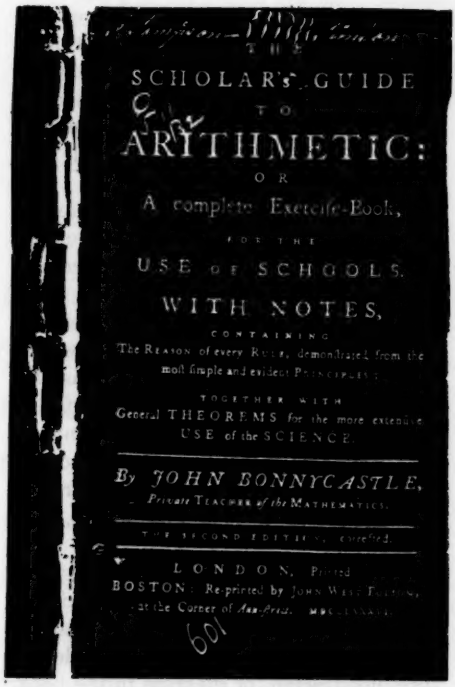
Arithmetic problems 175 years ago

by Cecil B. Read, University of Wichita, Wichita, Kansas

A commonly encountered criticism of present-day mathematics teaching is that we fail to take account of new developments; it is sometimes said that a mathematician of the seventeenth or eighteenth century could step into the modern classroom and be competent to teach any of the subject matter. Without the necessity of agreement or disagreement with this point of view, it should be made clear that the statement does not imply (as sometimes seems to be assumed) that our textbooks and classroom presentations have not changed, or that there has been no revision of content material.

The actual situation might be illustrated by consideration of a course generally considered as elementary, namely arithmetic. This would eliminate from our consideration questions of changes resulting from a combination of plane and solid geometry, introduction of new material, such as the concept of sets in an algebra course, inclusion of simple numerical trigonometry in an algebra or geometry course, "fused" courses, and the like.

The author has in his personal library an arithmetic book published in London in 1780 and reprinted in Boston in 1786. Comparison with a few other arithmetic books of that period indicate that it is typical of such texts at that time. The title itself is intriguing. (See the accompanying photocopy.) Apparently, brevity of title, as evidenced by this and contemporary works, was not considered meritorious.


THE
SCHOLAR's GUIDE
TO
ARITHMETIC:
OR
A complete Exercise-Book,
FOR THE
USE of SCHOOLS.
WITH NOTES,
CONTAINING
The REASON of every RULE, demonstrated from the most simple and evident PRINCIPLES:
TOGETHER WITH
General THEOREMS for the more extensive USE of the SCIENCE.
By JOHN BONNYCASTLE,
Private TEACHER of the MATHEMATICS.
THE SECOND EDITION, corrected.
LONDON, Printed
BOSTON: Re-printed by JOHN WEST, ...
at the Corner of ... Street. MDCCLXXXVI.
601

The range of content of this text seems much greater than is the case with current books. The first statement of the book is: "ARITHMETIC is the art of computing by Numbers, and has five principal or fundamental rules for its operation: viz. Notation, Addition, Subtraction, Multiplication and Division." After these topics are presented, there is a discussion of compound addition, subtraction, multiplication, and division. Proportion is introduced under the headings "The rule of

three direct" and "The rule of three inverse," and is followed by compound proportion. Among the topics not normally found in present-day arithmetics is "Tare and Trett," defined as "practical rules for deducting certain allowances which are made by merchants and tradesmen in selling goods by weight." (Trett is defined as an allowance of 4 lb. in every 104 lb. for waste, dust, etc.)

Topics which we currently would find in business mathematics (or even a college course in mathematics of finance) are treated in this book, in some cases quite extensively. Examples are simple and compound interest, commission, brokerage, buying and selling of stocks, discount, equation of payments, barter, single and double fellowship, and annuities.

There is, as might be expected, extensive treatment of fractions (both "vulgar" and decimal). Arithmetical and geometrical progressions, permutations, and combinations are included—topics we would more likely currently find treated in an algebra text. Powers as high as fifth power, and extraction of roots as far as ninth root are included. Problems involving exchange include many European currencies, for example, those of France, Venice, Spain, Portugal, Hambro, and Prussia.

Even a casual examination of the book reveals many points on which it differs from arithmetic texts commonly used at the present time. There is not a single picture, illustration, or diagram in the book. One is impressed by the extensive use of very large numbers. Numbers involving twelve to fifteen digits are not at all uncommon. Again, there is little tendency to have answers to problems come out as integers or simple fractions. In fact, the majority of the problems do not, as the student says, "come out even." This may have some merit, for it seems that present-day students think it almost axiomatic that there is an error if the answer to a problem involves a fraction such as 187/514. In view of answers as they appear in many current texts, such a student can hardly be accused of failure to reason logically. (The engineer sometimes assumes the converse; if an answer to a problem encountered in his daily work is an integer, there is probably something wrong.)

Something of the "flavor" of the text may be shown by quoting some problems, which not only shed light on prices and living and working conditions of the times but offer an interesting contrast to problems in current arithmetics. With the exception of the old English letter "s," these are presented as they appear. Since the author has found that sometimes students find it interesting to try to work these problems of former days, answers in some cases have been included. These are the answers as printed in the text; there is no guarantee of their correctness.

It should be pointed out that the problems reproduced are not a random or a representative selection, rather they are those which attracted the attention of the author as he examined the text. They are selected either to illustrate a particular point or because they seemed unusual. From this point of view they, without question, may form a biased sample. Nevertheless, the fact that many of these or their counterparts are not found in contemporary texts may help to dispel the idea that mathematics, and particularly arithmetic, has been completely static for the last two or three centuries.

How many minutes are there since the birth of Christ to this present year 1776, allowing the year to consist of 365 days, 5 hours, 48 minutes, 58 seconds? Ans. 934085364

Divide 558001172606176724 by 2708630425. Ans. 206008604 24/270

Multiply 6247486495 by 27356.

If 24 lb. of raisins cost 6s. 6d. what will 18 frails cost, each weighing neat 3qrs. 18 lb?

Shipped for Barbadoes 500 pair of stockings at 3s. 6d. per pair, and 1650 yards of baize at 1s. 3d. per yard, and having received in return 348 gallons of rum at 6s. 8d. per gallon, and 750 lb. of indigo at 1s. 4d. per lb; what remains upon my adventure?

If 248 men, in 5 days of 11 hours each, dig a trench 230 yards long, 3 wide, and 2 deep; in how many days of 9 hours long will 24 men dig a

trench 420 yards long, 5 wide, and 3 deep? Ans. 283 194/207 days

What is the compound interest of 410 l. forborn 2½ years, at 4½ per cent. per annum; the interest payable half yearly?

Bought 40 gallons of brandy at 3s. per gall. but by accident 6 gallons are lost, at what rate may I sell the rest, with 8 months credit, and gain upon the whole prime cost, at the rate of 10 per cent. per annum?

Express 6/7 of a moidore in the fraction of a pound.

Extract the sursolid, or fifth root, of 3076828-21106715625.

A labourer was hired for 40 days, upon this condition, that he should receive 20d. for every day he wrought, and forfeit 10d. for every day he was idle; he received at last 2 l. 1s. 8d.; how many days did he work and how many was he idle?

How many oaken planks will floor a barn 60½ feet long and 33½ wide; when the planks are 15 feet long and 15 inches wide? Ans. 108

Suppose the sea allowance for the commen men to be 5 lb's of beef, and 3 lb's of biscuit a day, for a mess of 4 people; and that the price of the first is 2½d. per lb. and of the second 1½d. now, if the ship's company be such that the meat they eat cost the government 12 guineas per day, what must they pay for their bread per week? Ans. 35 l. 5s. 7½d.

Bought 120 oranges at 2 a penny, and 120 more at 3 a penny, and sold them all together at 5 for 2d. what did I gain or lose by the bargain?

Divide 4637064283 by 57606. Ans. 80496 11707/57606

How many barleycorns will reach around the world, supposing it, according to the best calculations, to be 8340 leagues? Ans. 4755801600

At half a guinea a week how many months board can I have for 100 l.

If when the price of a bushel of wheat is 6s. 3d. the penny loaf weighs 9 oz, what ought it to weigh when wheat is 8s. 2½d. per bushel? Ans. 6 oz. 13 dr.

How many dozens of candles, at 5s. 2d. per doz. must be given in barter for 3 cwt. 2 qrs. 16 lb. of tallow, at 37s. 4d. per cwt.?

The International Colloquium on Differential Geometry and Topology in Zürich and the celebration of the fiftieth anniversary of the Swiss Mathematical Society

by Laura Guggenbuhl, Hunter College, New York, New York

At the end of June, 1960, the Swiss Mathematical Society celebrated its fiftieth anniversary at the Swiss Federal Institute of Technology and at the University of Zürich. In connection with the celebration, an International Colloquium on Differential Geometry and Topology was held under the chairmanship of H. Hopf. The Colloquium was sponsored by the International Mathematical Union and attended by more than one hundred mathematicians from many different parts of the world.

After a few brief opening remarks on Monday morning, June 20th, by Professor Hopf, the program began with an address, "Vector fields on spheres and allied problems," by R. Bott. The program ended on Friday afternoon, June 24th, with an address, "Mappings of complex analytic manifolds," by S. S. Chern. At other times during the week, lectures were presented as follows: "Convexity in Grassmann manifolds," by H. Buseman, "Homotopy and homology," by B. Eckmann, "Charakteristische Klassen und ihre Anwendungen," by F. Hirzebruch, "Einige neuere Ergebnisse in Differentialgeometrie und Topologie," by H. Hopf, "Transformations des variétés à connexion linéaire et des variétés riemanniennes," by A. Lichnérowicz, "A survey of cobord-

ism theory," by J. Milnor, "Quelques notions de base de la théorie des variétés différentiables," by G. de Rham, "The algebra of cohomology operations," by N. E. Steenrod, and "Stabilité topologique des applications différentiables," by R. Thom.

About fifty additional shorter papers were presented in two different sections. Papers of mathematicians from Bucharest, who were unable to attend the Colloquium in person, were read by others.

On Wednesday morning a memorial address was presented as a tribute to the late Professor J. H. C. Whitehead, who died in Princeton last Spring while on leave from the University of Oxford. Whitehead, son of the Bishop of Madras and nephew of the philosopher and mathematician, A. N. Whitehead, was born in India fifty-five years ago. Up until the day of his untimely death on May 8, 1960, he was planning to take an active part in the Colloquium. The address, which will be published in full in an early issue of *L'Enseignement Mathématique*, was presented by his friend and former student, P. J. Hilton.

On Thursday morning the Colloquium, upon the motion of Professor Hopf, voted to send a cable of congratulations to Maine to felicitate Professor Oswald Veblen on the occasion of his eightieth birthday the following day, June 24.*

The closing session of the Colloquium on Saturday morning, which was at the same time the opening session of the formal celebration of the fiftieth anniversary of the Swiss Mathematical Society, took place in the large new auditorium of the Physics Building of the Swiss Federal Institute of Technology. The lectures at this session—which were those by G. de Rham and H. Hopf listed above—were intended for an audience not necessarily restricted to specialists. Many additional mathematicians, friends and members of the Swiss Mathematical Society, attended this session.

Undaunted by threatening skies, members embarked on Saturday afternoon for a delightful cruise on the beautiful Lake of Zürich. One could not ask for a more ideal setting for a friendly opportunity to renew discussion of some points which had come up in the earlier days of the week and to meet members of the Colloquium in this somewhat less formal atmosphere. The ship sailed along the northern shore of the lake past beautiful lakeside villas, homes in the hills, and several poetic and historic spots, to Rapperswill, and then back to Bürkliplatz along the south shore. Refreshments were served along the way.

The final meeting of the formal part of the program was held on Sunday morning, June 26th, at the University of Zürich. Professor Jecklin, president of the Swiss Mathematical Society, presided, and a number of musical selections were rendered by a quartette of wind instruments. An address, "Mathématiques et Mathématiciens en Suisse (1850–1950)," was presented by Professor Plancherel, former president of the Society. Following the address, the rarely awarded distinction of honorary membership in the Swiss Mathematical Society was conferred upon G. de Rham of Lausanne, J. Leray of Paris, and H. Whitney of Princeton.

The proceedings of the week's celebration were brought to a close at a banquet at the lakeside Kongresshaus on Sunday afternoon, June 26th. Professor Eckmann, vice-president of the Society, served as a most cordial toastmaster, while representatives of many mathematical and educational organizations brought greetings and best wishes.

Hotel accommodations for those who participated in the week's activities were most comfortable and convenient. In most cases, members stayed in groups at various hotels in the central part of Zürich, many within walking distance of the Swiss Federal Institute of Technology and the

* Later in the summer, members were deeply saddened to hear of the death of Professor Veblen on August 10, 1960.

University of Zürich. In other cases, a short trolley ride would take one to the place where meetings were being held. Opportunities for discussion were ample and cordial, not only during the regularly scheduled discussion periods after the talks on the program, but also at the informal teas, coffee breaks, and social gatherings during the week. Others who had an active part in the preparations for the meetings were the secretary-treasurer of the Swiss Mathematical Society, Professor de Siebenthal, and Professor Künzi. C. J. Curjel served as secretary of the Colloquium.

Many of those who made such generous contributions of time, thought, effort, and financial resources to the great success of this celebration and international Colloquium remained completely anonymous, and the mathematicians who came from scattered and distant parts of the earth's surface were quickly made to feel thoroughly at home.

The Colloquium did not pass unnoticed in the daily press. No doubt a number of visitors were agreeably surprised by the extent of co-operation of the city's leading newspaper, the *Neue Züricher Zeitung*. This newspaper found a large amount of space for three fairly technical articles: "Mathematische Fragestellungen im Grossen, Internationales Kolloquium über Differentialgeometrie und Topologie," by Beno Eckmann, "Aspekte der heutigen Mathematick," by Rudolf J. Hoesli, and "Die Schweizerische Mathematische Gesellschaft, 1910–1960," by Eduard Fueter.

The Commentarii Mathematici Helvetici was well represented by Professors J. J. Burckhardt, A. Pfluger, G. de Rham, M. Plancherel, P. Finsler, H. Hopf, F. Fiala, G. Vincent, H. Jecklin, and others. The presence of Professor Nevanlinna of Helsinki brought to mind the approaching International Congress of Mathematicians which will be held in Stockholm in 1962.

It was a rare and deeply appreciated privilege to be at the celebration of the fiftieth anniversary of the Swiss Mathematical Society and at the 1960 International Colloquium on Differential Geometry and Topology in Zürich.

A brief history of mathematics in rhyme—the early Greeks to the seventeenth century

by Marcella B. Krueger, Senior High School, Chetek, Wisconsin

THE GREEKS, B.C.

Through the math'matic maze we go
To clinch the names we ought to know:
The first we spot is Father Thales
Who in 600 made the dailies—
With pyramids and ships at sea
He put the "why" in geometry.

Then Pythagoras in 540, too,
Made Bab'lonian triples look like new.
Zeno, 450, watched the turtle's track
And wondered if he'd e'er get back.
He stopped the arrow in its flight—
Presaged math concepts infinite.

Democritus, 410, went the limit—
Cut the pyramid to see what's in it!
Socrates' ace, Plato, was no fool—
He founded the Athenian School.
And Eudoxus (370) saved the Old Greek's face:
Proved the square root of two in its proper place.
His incommensurable family was such a find
It tickled the wits of Dick Dedekind.

While quadrature, trisection, and duplication
Seemed to demand a new creation—
Hippocrates had, in 440, without distortion
Reduced "duplication" to a duoproportion.

While with cylinder, torus, and right circular cone,
Archytas proved in 400 how it ought to be done.
Menaechmus, 350, would be no man's debtor
And invented the conics to prove it much better.

In 300 Euclid recorded the best,
His own touch of genius cohered all the rest,
And he gave to the world a most logical form
To stop all the dolts at his pons asinorum.

Earth measurements settled, minds pondered the moon,
Aristarchus of Samos, Copernicus' boon,
In 287, not to be outdone,
Actually measured the distance of moon to the sun.

Archimedes of Syracuse, great teacher and sage,
Proposed mathematics too keen for his age;
For others less gifted had made him a clown
By playing the scandal of King Heiron's crown.
His friend Eratosthenes, a Beta by birth,
Developed the "sieve" and measured the earth.

Apollonius followed, a third giant brain,
Linked conic sections to the math'matic train;
Ellipse, hyperbola, parabolic persuasion
Explicitly determined by an equation.

The 200's ended, but 140 found
Hipparchus' eyes on the stars, his feet on the ground;
For trigonometry's growth, as the history records,
Followed 360 degrees and his table of chords.

The Greeks, A.D.

T'was Heron, near 50, who with Egyptian air
Gave Greek geometry a practical flair;
He measured everything under the sun,
Took square roots to leave nothing undone
While Diophantus' untraditional manner
Raised algebra's "syncopated" banner.

Menelaus' studies of chords and spheres
Were preserved by the Arabs, sound through the years.
In 150 astronomy's great Almagest
Put Claudius Ptolemy up with the rest.
Though Copernicus and Kepler replaced his views,
His parallel postulate still is the news.

Reflecting the date of Euclid, the great
Pappus arrived to elucidate.
For two centuries the commentators tried
To keep the high flame of math'matics alive:
Papa Theon and daughter Hypatia, too,
While Proclus studied Eudemus anew,
And Metrodorus (500) in epigram
Is the best Greek source we have at hand.
Boethius, the Roman, and Simplicius tend
To the final chapter of a good book's end.

The Hindus

But the song couldn't die, it just
Moved East to the Aryabhatas.
For Hindu trig and astronomy fine
To the half-chord they aptly gave the name sine.
Though zeros and numerals were used by a few,
The place value of ten was certainly new.

Brahmagupta's quadratics new beauty achieved,
With pulverizer method and roots negative.

In operations arithmetic Mahavira was sly:
To divide by a fraction, invert, and multiply!
He handled quadratics as well as the rest,
But the square root of a minus he couldn't digest.

The algebra of Bhaskara was fine, we are told
His proof in geometry was simply, "Behold!"
Ramanujan, in 1880, self-taught Hindu fine,
Made interesting talk of his number line.

The Arabs

The Arabians' chief bid to fame
Was made in the translation game.
Al Khowarizmi moved along its way
The best in algebra of his day.
His algorisms changed the lists
Of European abacists.
His ciphers and zero won him high fame,
He was first to call "al-jebr" by name.

Tabit ibn Qorra, true to the trends,
Translated Euclid and his friends;
While Abu'l-Wefa, translator, too,
Added tangent function to tables new.
Abu Kamil, Al Khowarizmi rewrote,
Al-Karkhi added a place-value note.

Omar Khayyám, most original Persian,
Geometrically solved the cubic equation.
Nasir ed-din found, and none too soon,
That trig has some merits of its own.
So with the tables of Ulugh Beg
We turn the page on the Arab's day.

Europe, 500–1600

The Dark Ages of Europe found culture's retreat
To the Hindus, Arabs, and the few who could eat.
We've mentioned Boethius, the Roman who
Wrote mathematics texts in 500, too.
Bede, 700, the venerable churchman,
Worked on the calendar, computed by hand.

Alcuin, 800, to assist Charlemagne
Wrote puzzle problems of considerable fame.
Gerbert, 1000, French student in Spain,
Brought back the numerals, Hindu-Arabic by name.
Near the year 1000 the translators began
To bring Moslem culture to the wakening land.

Adelard of Bath, English monk, though 'twas risky,
Did a Latin translation of Al Khowarizmi.
And Euclid's Elements again share the honor
In 1150 with Gherardo of Cremona;
He Latinized ninety of Arabia's best,
Among which was Ptolemy's great Almagest.

Fibonacci of Pisa revived algebra still
From Al Khowarizmi, through Abu Kamil.
In Liber Abaci his creative bent
Set him apart in his element.

Oresme with his co-ordinate location
Saved the fourteenth century from obliteration.
Aquinas and Archbishop Bradwardine
Wrote of continuous, discrete, large, and superfine.

The fifteenth century saw the flame
Of Greek mathematics revive again.
The printing press and a new nation
Built growing trades and navigation.
Regiomontanus and Pacioli wrote
Math texts of the "Summa" note,
With many plans of multiplication
And even traces of syncopation.

Filippo Calandri's printed essay
Showed long division the modern way.
The Cardano-Tartaglia altercation
Aided the cubic situation.
But the quartic, too, a jinx must be
Until Ferrari found the key.
Vieta, who left as Descartes came,
Was the first of the French to the hall of fame.

Reviews and evaluations

Edited by Kenneth B. Henderson, University of Illinois, Urbana, Illinois

BOOKS

Applied Boolean Algebra, An Elementary Introduction, Franz Hohn (New York: The Macmillan Company, 1960). Paper, xx+139 pp., $2.50.

During these times of upheaval in the mathematics program there is danger that too much naïveté may become a part of secondary-school classroom practices. People with no more background than that in the textbook they are using do not hesitate to go into topics of current mathematical interest with their students. Such teachers speak of taking up "set theory" and "Boolean algebra" with their students. This reviewer has often wondered whether such teachers ever help their students understand the reason for the importance of such theories to mathematicians. But the wondering is tempered with sympathy. Where would busy teachers find material that they can read and understand and study to acquire some depth in these theories? Insofar as set theory is concerned, there are many excellent references; but there has been no good elementary treatment of Boolean algebra until the appearance of Professor Hohn's little book. In it one finds a careful, understandable development of the theory as well as many applications.

The book begins with a discussion of mathematical models and the relationship between them and the physical world. This introduction sets the stage for the formulation of the postulates for a Boolean algebra. The postulates are accompanied by comments that serve as explanations and interpretations. Only those consequences—theorems—of the postulates are developed that will be useful in the applications that follow. Professor Hohn next proceeds to build a mathematical model of combinatorial relay circuitry. It turns out that "we may regard the set of all switching functions of n circuit variables as a Boolean algebra." A following chapter presents Boolean algebra as a model of propositional logic. This chapter ends with a discussion of logic computers. There are throughout clear explanations on an elementary level so that a reader gains an excellent knowledge of the implications of the theory. The third chapter treats the Boolean algebra of the subsets of a set, and the fourth and final chapter takes up the problem of simplifying, or minimizing, representations of Boolean functions.

Scattered through the chapters are exercises that enable a reader to see whether he has mastered the theory presented in preceding pages. These exercises seem to be carefully chosen to have a nice balance between further extension of the theory and applications of the results in the book. Anyone who studies Professor Hohn's book will be able to do the sort of teaching that leads students to an understanding of the importance today of Boolean algebra and to an appreciation of the variety of areas of mathematics and electronics in which Boolean algebra has an application.—*Myron F. Rosskopf, Teachers College, Columbia University, New York, New York.*

The Arithmetic of Computers (TutorText), Norman A. Crowder (Garden City, New York: Doubleday and Company, 1960). Cloth, 472 pp., $3.95.

Adventures in Algebra (TutorText), Norman A. Crowder and Grace C. Martin (Garden City, New York: Doubleday and Company, 1960). Cloth, 350 pp., $3.95.

One might characterize these books as "machineless teaching machines" because they represent one type of programmed text materials Those familiar with the Skinner type of teaching-machine program will find that the "program" of these books is structured differently from the more familiar sequence of completion items through which the learner passes regardless of the correctness of his responses. In the TutorText series, the learner's experience is in the following form. He first reads a short passage of text. This reading is followed by a pertinent question for which two or more answers are supplied (i.e., a multiple-choice question). Beside each answer is a page number. The reader chooses one of the answers and turns to the page whose number appears beside the answer. On the new page, he is told whether he has chosen correctly. If his response was incorrect, the nature of his error is pointed out and discussed and he is referred to the page from whence he came to try again. If his response was correct, the next step of the learning program appears on the new page, sometimes preceded by material to reinforce the correct response to the previous item with further explanation or discussion.

The rationale for Crowder's method ("intrinsic programming") focuses upon the use of feedback from the learner to control the communication process by which information is transmitted to the learner. By "branching" on the learner's response, the program automatically speeds the learner through those areas with which he is already partially familiar or that he can learn readily, and causes him to

move at a more deliberate pace through those areas where he needs to work more carefully. Furthermore, when difficulty arises, appropriate corrective action is taken for the particular type of error that has occurred.

One can take issue with the plausibility of the alternatives to the correct response in some of the items in the two books. Whether this is a really serious objection, however, is a complex issue, in view of the psychological dictum that the probability of a wrong response should be kept low at all costs. Nevertheless, one is led to doubt the assertion on the jacket blurb, "A TutorText makes it impossible for you to get to the end of the book without understanding it."

The Arithmetic of Computers deals with systems of numeration related to our everyday decimal positional notation. Bases 2 and 8 receive primary attention, due to their importance in working with electronic digital computers.

The first third of the book treats the topic of exponents and powers of numbers and provides an excellent elementary introduction to that topic. The middle third of the book introduces the octal (base 8) and binary (base 2) numeration systems and examines related problems, such as conversion from one base to another. The final third of the book treats binary arithmetic from a digital-computer-oriented point of view.

Chapter summaries are supplied at the end of each chapter, and self-tests are included for each chapter (using an ingenious answer keying system that enables the student to check his answer to one problem without compromising the other answers in the key). These aids seem to be especially necessary for this type of treatment. On the whole, this book seems to approximate very closely the ideal of self-teachability.

Adventures in Algebra provides an interesting little excursion into elementary algebra of the type studied in a garden-variety high school mathematics course. It seems safe to assume that the book is not intended by the authors as a text for a standard high school course in algebra; hence it would seem pointless to dwell on the shortcomings of the book for that use.

The main theme of the book is the generalization of the concept of number from the natural numbers to the integers, the rationals, and beyond. However, nothing resembling a systematic treatment is undertaken.

Probably the strongest point of the book is the manner of approaching the proofs of three theorems concerning, respectively, the nonexistence of a largest prime, the formula for the sum of the first n natural numbers, and the irrationality of the square root of two. The learner is led deftly along the thread of proof in such a way that he is forced into a degree of personal involvement in the proof beyond that possible under any other known method of development. This alone is sufficient to make the book worthy of interest.

The claim of true self-teachability for the entire book would seem to be a bit overoptimistic. Nevertheless, the learning per unit time invested by the learner (and by the teacher, for that matter) would seem to be greater than for other media. (This could even turn out to be embarrassing, as the teacher who has supplied reasonably well-constructed self-teaching materials to a bright student can testify.) The student may finish working through the text before the teacher has time to plan what to start him on next.

If one were to choose between the two books, *The Arithmetic of Computers* would seem to be the far better buy. It evidences much better planning and organization; it is a more thorough treatment of its subject matter; and it seems to have a more clearly defined potential use. It can be used for enrichment purposes with little or no teacher effort, while it is difficult to see just how the teacher could make effective use of *Adventures in Algebra* in most school situations. The latter book also lacks the important chapter summaries and self-tests.

Many teachers will find these TutorTexts represent one of the most exciting ideas since the printed page itself. The development of adequate programmed text materials may prove to rank in importance well ahead of, for example, the development of audio-visual aids as a milestone in educational progress. If, when judged by traditional textbook standards, these two programmed texts do not achieve perfection in all respects, they at least merit the attention of the teaching profession.—*George A. Robinson, Hughes Aircraft Company, Fullerton, California.*

Elementary Statistics, Paul G. Hoel (New York: John Wiley and Sons, Inc., 1960). Cloth, vii+261 pp., $5.50.

According to the author, "This book is designed for a one-semester course for the student whose background in mathematics is limited to high-school algebra." Such a student, the author believes, should be taught through the simple classical ideas of sampling very thoroughly, rather than by attempting some of the more sophisticated modern notions. Some persons will argue that many of the modern ideas are as simple as the classical ideas, which at one time, no doubt, were thought extremely sophisticated.

In spite of the limited mathematical background assumed, the author has packed a good deal of solid information and some feeling for the underlying statistical ideas into this book. The author is a mathematical statistician rather than a member of those groups who produce the majority of elementary statistics books—the economists, the psychologists, and so forth. In spite of this, or because of this (according to one's point of view), the fundamental ideas are clearly related to their applications.

There are chapters on the nature of statistical methods, the description of sample data, probability, theoretical frequency distributions, sampling, estimation, testing hypotheses, correlation, regression, the chi-square distribution, nonparametric tests, analysis of variance, and time series and index numbers.

Several features add to the usefulness of this book. There are a considerable number of good problems at the end of each chapter. Answers to odd-numbered problems are given at the end of the book. There are more tables of statistical distributions presented than in most books on elementary statistics.—*D. L. Burkholder, Department of Mathematics, University of Illinois, Urbana, Illinois.*

The Higher Arithmetic, An Introduction to the Theory of Numbers, H. Davenport (New York: Harper and Brothers, 1960). Paper, 172 pp., $1.35.

The subtitle seems to convey a better description of this book than does the title. Perhaps the British (the author is a professor at the University of London) refer to number theory as higher arithmetic. Some historical material is included. Information about the men who formulated the major conjectures and their proofs and the dates of these events are an interesting part of this text.

The early development of the natural numbers, with some comments on the integers and rational numbers, features the associative, commutative, distributive, and order axioms, and mentions the word "set." This brief review of more modern ideas and rare mention of them later in the text is in vogue; and it is perhaps the best way to lead into the author's subject.

To discuss divisibility and primes, the author gives a very clear, informal description of the method of mathematical induction. Then on page 16, he makes this rather surprising statement, ". . . it is possible to debate whether the principle is in the nature of a *definition* or a *postulate* or *an act of faith*." With the addition of the well-ordered principle to his list of axioms, one can give a short indirect proof of the principle. If one does not choose to add this principle, he of course would postulate the principle of induction and then prove the well-ordered principle if desired.

No exercises are included in the book. This would be a serious deficiency for use in a classroom as well as for independent study.

In the introduction, the author recognizes a certain amount of mathematical maturity is required to read this book. Even superior secondary-school students would have to proceed at a deliberate rate. The concept of limit (page 92) and the use of new symbolism seem to suggest college-level placement. An informal type of proof where one doesn't dot each i and cross each t is utilized, leaving the reader to fill in the details. The proofs are made much clearer by a discussion of what is to be shown and how this is to be accomplished. Numerical illustrations are given occasionally to illustrate the principles and show how they can be applied.

Among the many instances where this book is more complete and detailed than others, we find the concept of linear dependence used to prove that the highest common factor of a and b can be represented as $ax - by$ where a, b, x, y are natural numbers. Congruences of degree n are discussed and solved by the factoring method which relates nicely to the solution of polynomial equations of degree n in a college algebra course. It is further shown that such congruences cannot have more than n solutions in contrast to having exactly n solutions as an equation.

The reviewer found the section on continued fractions, which is sometimes not included in the newer texts, interesting. This section shows another application of the Euclidean algorithm. A very complete description of the types of integers that can be expressed as the sum of two, three, or four squares is included. A clear and brief discussion of Diophantine equations is given. The reviewer felt that perhaps the section on quadratic forms could have been replaced by material concerning indeterminant problems.

This book would provide a teacher with a clear and readable summary of the research on number theory. It would be a valuable text for a college-level course if it were supplemented with appropriate exercises.—*Charles R. Stegmeir, Lyons Township High School, Lyons, Illinois.*

Introduction to Modern Algebra, Official Textbook for Continental Classroom, John L. Kelley (Princeton, New Jersey: D. Van Nostrand Company, Inc., 1960). Paper, x+338 pp., $2.75.

This book is concerned with axioms for the real numbers, elementary concepts of sets and statements, vectors as ordered pairs and ordered triples of real numbers, complex numbers as ordered pairs of real numbers, the algebra of vector spaces, and matrices. Applications of each concept are made to a variety of problems, including many problems normally considered in high school. The point of view is that of abstract algebra; the scope of topics includes many items that high school teachers think of as geometry. In this sense, there is an emphasis upon the interrelations of algebra and geometry.

Theorems and proofs are emphasized throughout the book. Figures and illustrative examples are used freely. Almost all of the 47 sections have individual sets of problems—some routine and a few sophisticated. Solutions for nearly all of the problems and some commentary upon the text are given in a *Manual* which is restricted in its circulation primarily to teachers using the text in their classes. This teacher's manual and a small student's manual are also published by D. Van Nostrand Company.

The book is designed for a first-year course in a liberal arts college program or for high school students in an accelerated program. Teachers may use the book to increase their understanding of several of the newer concepts recommended for inclusion in school mathematics curricula. In doing this, however, teachers should recognize that the presentation and often the vocabulary in the book may be quite different from that needed in secondary schools.

The book should properly be considered as a

preliminary edition with very many "rough spots" arising from the necessity of production on a strict schedule. Several improvements have been made by the author in his TV presentations (especially in Sections 12, 23, 24, and 30); others will be made as soon as there is an opportunity to do so. Also, several items which seem "unfortunate" in relation to the needs of teachers may appear acceptable in an introductory course for college freshmen or in a refresher program for people whose mathematical training has lain fallow for several years.

Teachers using this book as a basis for "modernizing" their own courses are urged to modify the treatment in at least the following respects: (1) Make a careful distinction between a function and its graph. (2) Define and use equivalent vectors. (3) Distinguish between points and vectors to avoid dual use of symbols, as in Definition 18.3. (4) Emphasize the domain under consideration at all times and in particular for the solution of a quadratic equation over the set of real numbers on p. 175, over the set of complex numbers on p. 219. (5) Recognize that a cone has two nappes (Section 28). (6) Replace the term "complex multiplication" by "multiplication of complex numbers" (p. 202 ff.). Similarly, replace the term "complex equations" by "equations with complex coefficients."

In its present form, the book is hard reading for teachers with modest training, but worthy of serious study for mathematics teachers with at least average training. High school students with special training or interest will find many topics interesting. The author has a fine sense of humor. The style of exposition is frequently easy going. Each reader will find many problems of special interest.—*Bruce E. Meserve, Montclair State College, Upper Montclair, New Jersey.*

Modern Mathematics, An Introduction, Samuel I. Altwerger (New York: The Macmillan Company, 1960). Cloth, xii+462 pp., $6.75.

The phrase "modern mathematics" can mean two distinctly different things. It can mean mathematical topics developed in the twentieth century or it can mean traditional mathematics developed from a twentieth-century point of view. In recent years there has unfortunately been too much trafficking in "modern" topics and "modern" terms. The result in too many instances has been a distorted picture of modern mathematics from either of the above points of view.

This reviewer believes that it is high time that textbook writers be called to task for any misuse of either words or topics. Since there are essentially no mathematical topics developed in the twentieth century included in this book, it must be assumed that *Modern Mathematics, an Introduction* is to be judged as a treatment of traditional mathematics from a twentieth-century point of view.

Certainly some of the salient features of such a development should be clarity of ideas, careful use of language, a feeling for the abstract nature of mathematics, and insight into the deductive and sequential nature of mathematics. It seems to this reviewer that the book falls short in all of these features. Here are some examples which illustrate this shortcoming.

Page 5: "A system involving five as a base has only five distinct symbols." (Careful use of language?)

Page 8: The five fundamental assumptions of arithmetic are given—*no* mention of order is made. (Sequential nature of mathematics?)

Page 35: We are given a proof of a theorem with one of the reasons cited being the "postulate of identity," yet this reviewer could find no such postulate. (Deductive nature of mathematics?)

Page 49: We are told that "Irrationals with different radicands cannot be added or subtracted." Isn't $\sqrt{2}+\sqrt{3}$ a real number? (Careful use of language?)

Page 60: We are told, essentially, that all there is to Dedekind's definition of a real number is the construction of two disjoint sets. The real problem is of course to show that these cuts behave like numbers—but no mention of this. (Clarity?)

Page 106: We are told that "A set of paired values is known as a function." (Careful use of language?)

Page 151: We are told that from the development of non-Euclidean geometry came "the notions of the relativity [sic!] of postulational thought." (Abstract nature of mathematics?)

Page 171: We are told that "$p \wedge q$ is true only when p is true and q is true." Yet it seems that "only when" depends upon implication which is first presented on page 175. (Sequential nature of mathematics?)

Page 214: We are told that, given $x=10^y$ and $x=e^y$, "to solve these for y, we need both a *name* and a *symbol.*" It would, indeed, be comfortable if solutions depended only on names and symbols. (Clarity?)

Page 323: We are told that "The rate of change at a point is often referred to as the *instantaneous rate of change* which is suggestive of the intent although the time factor is not necessarily in the picture." (Abstractness?)

There is also a constant shifting of point of view which is confusing to the reader. For example, the trigonometric ratios are introduced as functions, then they are treated as numbers, then they are treated as functions, and then they are treated as numerical constants.

It seems to this reviewer that it is possible that a series of very beautiful and inspired lecture notes was transcribed to make this text. But it remains a fact that there is a substantial distance between a set of lecture notes—however inspired—and a good text.—*Morton R. Kenner, Southern Illinois University, Carbondale, Illinois.*

The Number Story, Herta Taussig Freitag and Arthur H. Freitag (Washington, D.C.: The National Council of Teachers of Mathematics, 1960). Paper, 76 pp., 85¢.

Through the ages, man has gradually developed number ideas, he has observed, noted relationships, and endeavored to prove conclusively the general nature of ideas developed. In this monograph, the authors have presented some of the high lights in this story of the evolution of number from the first steps taken by prehistoric man to the mathematics of our present day. The historical development of the growth of number notions in regard to natural numbers, fractions, zero, and negative numbers, irrational numbers, and complex numbers is briefly presented. The authors also give a somewhat intuitive mathematical development of this structure.

It is judged by this reviewer that many teachers of secondary mathematics and many capable students of mathematics will find this monograph readable and thought provoking. Some specific examples of topics included seem in order here. One section is concerned with highlighting certain aspects of the historical development of the idea of recording numbers. Many teachers are generally familiar with the symbols and characteristics of several nonpositional and positional systems that are described. Some attention is given to base systems other than ten with reference to the use of a binary system of numeration in today's society. A description of the historical development of methods of computation begins with the use of the abacus as a type of counting machine. Early algorithms are described, such as halving and doubling, casting out nines or elevens, gelosia or lattice multiplication, the galley or scratch method, and Napier's bones as a device for multiplying, dividing, and extracting roots. Mention is made of the development of logarithms and of the idea of the automatic computer. Most readers of this monograph will know of other sources which also present these early means of computing. Ideas from the theory of numbers and some number peculiarities are presented. A procedure is developed for converting any nonterminating, periodically repeating decimal into a common fraction or mixed number, and an interesting discussion of converting irrationals into continuing fractions is included. Some surprising connections between π and the integers as found in the form of nested roots, infinite series, continued fractions, and infinite products are mentioned for the interest of the mathematically curious. The authors appropriately comment that the more abstract mathematics has become, the more widely its ideas have been applied.

Illustrations are sprinkled throughout the monograph in the development of ideas selected for highlighting. Some readers will think of other ideas that might well have been included, while other readers may feel a need for a fuller development of some of the ideas that are included. Nevertheless, the reader will probably receive inspiration from this brief glimpse into the beginning and extension of the concept of number to the more rigorous and generalized mathematical reasoning of today. This reviewer felt that a selected bibliography might well have been included to encourage further reading into the historical background for the various ideas mentioned in the monograph.—*Frances Flournoy, Department of Curriculum and Instruction, University of Texas, Austin, Texas.*

Supplementary Units in Contemporary Arithmetic and Elementary Algebra, Edwin I. Stein (Princeton, New Jersey: D. Van Nostrand Company, Inc., 1960). Paper, 75 pp., $1.

Are you looking for materials to help you teach basic topics in contemporary mathematics to your secondary-school pupils? Do you need suitable units on systems of numeration? Graphs of inequalities? Sets and logic? This 75-page booklet was designed to serve just such needs.

If your experience with contemporary topics in mathematics has been as a college student, you will be puzzled by the selection of materials suitable for students as young as seventh graders. Stein's booklet helps you with your selection of appropriate topics and shows you how to present the mathematical concepts in a simple and understandable way. He has chosen only topics basic to experimental programs and generally recommended by committees on mathematics education.

The author suggests that *Supplementary Units in Contemporary Arithmetic and Elementary Algebra* is most valuable when incorporated in an elementary algebra course. It may, however, be used on the seventh, eighth, or ninth-grade levels, either integrated with the traditional content or used as supplementary topics in a regular course. It can accompany any textbook. You will find the booklet easy to use, with a good table of contents and index, an attractive arrangement of materials on the page, and the eye-catching device of heavy, black type for important words.

There are three units: Number and Its Properties; Open Sentences, Equations and Inequalities, Sets and Logic; Graphs. In the first unit, students are introduced to the various number systems from natural to complex. The practice exercises, and there are an abundance of them, help pupils learn to recognize whether selected numbers do or do not belong to the various systems. Further explanation helps students recognize even, odd, prime, and composite numbers. As motivation, the Sieve of Eratosthenes is demonstrated.

An arresting section is the one on systems of numeration. After explaining how to change from one base to another, Stein follows with computations not only in the binary and duodecimal system, as are usually included, but also computation in the ternary, quaternary, quinary, senary, septenary, octonary, and nonary

(*Continued on p. 377*)

• TIPS FOR BEGINNERS

Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan, and William C. Lowry, University of Virginia, Charlottesville, Virginia

Finding illustrative examples and exercises

by Dunstan Hayden, The Priory School, Washington, D.C.

The alert teacher knows it is dangerous to construct examples and exercises purely at random. The results will frequently lack variety and interest and fail to manifest all the various possibilities. What is needed are formulas which will generate problems or examples in such a way that the teacher can *predetermine* the characteristics in question. A familiar example is the construction of quadratic equations in one variable. By working from the discriminant, b^2-4ac, one can select coefficients which will make the roots real or complex, rational or irrational, and so on.

It sometimes proves unexpectedly difficult to create such formulas. The author, after repeatedly reaching the threshold of exasperation, has taken time to dispose of some of his own *bêtes noires*. Fellow-sufferers may find the results of some interest.

Situation 1. During algebra class you wish to make up and exhibit several linear equations and their graphs. You are not interested in their slopes. How can you do this with a minimum of computation?

A solution. Of course there are many ways of making up linear equations and their graphs. The following method has two things to recommend it: (1) it is exceedingly rapid, and (2) it increases accuracy in making graphs on a blackboard quickly.

Select two numbers, X and Y, as the x-intercept and the y-intercept. Draw the graph. The equation whose graph goes through $(X, 0)$ and $(0, Y)$ may always be written in the form $Yx+Xy=XY$.

Example. Select -3 and $+7$ as x-intercept and y-intercept, respectively, and draw the graph. The equation is $7x-3y=-21$. Such an approach involves only one computation, finding the product XY.

Situation 2. Students are learning to graph parabolas and you want their exercises to include some "matching" parabolas like those shown in Figure 1. You wish to find equations for such curves.

A solution. In Cases I and II of Figure 1, the upper parabola has been rotated 180 degrees about the x-axis. If the original equation is $y=ax^2+bx+c$, then the equation of the new curve is $y=-ax^2-bx-c$. This should be no great surprise; for if (x, y) is a point on the first curve, then $(x, -y)$ will be a point on the second,

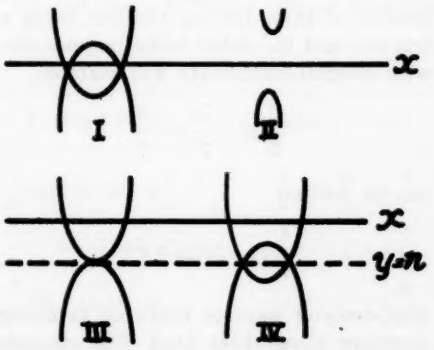


Figure 1

and (x, y) is the image of $(x, -y)$ as reflected in the x-axis.

In Case III, the parabola $y=ax^2+bx+c$ is rotated about the line tangent to its vertex. Remembering that the vertex always has co-ordinates

$$\left(\frac{-b}{2a}, \frac{-b^2+4ac}{4a}\right),$$

one may deduce—the argument is omitted—that the two equations will be related as follows;

$$y=ax^2+bx+c$$

and

$$y=-ax^2-bx-\frac{b^2}{2a}+c.$$

For example, if $y=x^2-6x+8$ is one of the equations, the other is $y=-x^2+6x-10$.

In Case IV, the parabola is rotated about *any* line $y=n$. This is a generalization and includes the first three as special cases. The equations—again without proof—will be $y=ax^2+bx+c$ and $y=-ax^2-bx-c+2n$. If $n=0$, we are back to Cases I and II. If

$$n=\frac{-b^2+4ac}{4a},$$

we are back to Case III. As a final example, let the parabola $y=x^2-8x+18$ be rotated 180 degrees about the line $y=3$. The new equation is then $y=-x^2+8x-12$.

Situation 3. Some polynomials with fractional coefficients can be written as the product of three factors, the first being a fraction and the other being polynomials with integral coefficients. For example,

$$\frac{2}{7}x^2-\frac{5}{7}x-\frac{3}{7}$$

can be written

$$\frac{1}{7}(x-3)(2x+1).$$

Suppose you want to make up factoring exercises along these lines. The example above is very simple because the denominators are all the same. You wish a method which will produce a diversity among the exercises. In particular, you want to be able to produce examples in which the three fractional coefficients have different denominators.

A solution. If we choose fractional coefficients at random, the resulting polynomial may not be factorable in the sense indicated above. If we start with a factored form and choose the fraction and the integral coefficients at random, then in the resulting product it is likely that two or even all three of the denominators will be the same. What we are seeking is a sufficient condition guaranteeing that the denominators (after simplifying) will all be different.

Suppose we adopt as a general schema

$$\frac{1}{a}(x+b)(x+c),$$

with a, b, and c distinct non-zero integers and $a\neq 1$. For a given number a, how are b and c to be chosen? We might make b prime relative to a, or a multiple of a, or a proper divisor of a, or at least have a factor in common with a; and similarly for c. If we want the final three denominators all to be different from 1, then neither b nor c should be a multiple of a. The reader may prove for himself that no combination of the other relations mentioned will infallibly produce the desired result. So we switch to a broader schema,

$$\frac{1}{a}(bx+c)(x+d).$$

Investigation of this product in the form

$$\frac{b}{a}x^2+\frac{bd+c}{a}x+\frac{cd}{a}$$

will show that the following is a sufficient condition that the three denominators, after simplifying, be all different and all different from 1.

Let b *and* d *be distinct proper divisors of* a; *and let each pair,* b *and* d, *and* c *and* a, *be relatively prime.*

TABLE 1

a	b	c	d	Factored Form	Product Form
6	3	5	−2	$\frac{1}{6}(3x+5)(x-2)$	$\frac{1}{2}x^2-\frac{1}{6}x-\frac{5}{3}$
−10	2	−7	5	$\frac{-1}{10}(2x-7)(x+5)$	$\frac{-1}{5}x^2-\frac{3}{10}x+\frac{7}{2}$
18	9	−11	2	$\frac{1}{18}(9x-11)(x+2)$	$\frac{1}{2}x^2+\frac{7}{18}x-\frac{11}{9}$
15	−3	2	5	$\frac{1}{15}(-3x+2)(x+5)$	$\frac{-1}{5}x^2-\frac{13}{15}x+\frac{2}{3}$

Table 1 shows a few examples constructed in accord with this condition.

Unfortunately, the examples produced by this method *always* have 1 or −1 as the numerator in the coefficient of the first term, and the largest denominator (in absolute value) is *always* divisible by the other two. If we adopt the more general schema

$$\frac{1}{a}(bx+c)(dx+e),$$

then the following is a sufficient condition that the three denominators all be different from 1 and from each other, that the largest (in absolute value) be not in every case divisible by the other two, and that the first numerator be not in every case 1 or −1.

Let a, b, c, d, *and* e *be distinct non-zero integers with* $a \neq 1$; *let* b, c, *and* d *be proper divisors of* a, *but* bd *not a multiple of* a; *and let each pair,* b *and* c, e *and* c, *and* e *and* a, *be relatively prime.*

Some examples are given in Table 2.

Situation 4. You wish to make up some problems in which the student must calculate the area of a quadrilateral, given the co-ordinates of the vertices; and you want some of the answers to be integral, some fractional, and some irrational. How can you select suitable vertices?

A solution. One way of calculating such areas is to draw a diagonal and then calculate separately the areas of the two triangles, perhaps using the determinant formula shown in Figure 2. A faster way is to use the second order determinant shown in Figure 3. (The formula in Figure 3 can be derived from the previous one by expansion of two of the latter, addition, and then some judicious factoring.)

To make the answer come out the way you want, choose numerical values for all co-ordinates except x_1 and x_3, put them into the determinant, and expand it. Then you can easily find numbers for x_1 and x_3

TABLE 2

a	b	c	d	e	Factored Form	Product Form
45	9	5	3	1	$\frac{1}{45}(9x+5)(3x+1)$	$\frac{3}{5}x^2+\frac{8}{15}x+\frac{1}{9}$
24	−3	8	6	5	$\frac{1}{24}(-3x+8)(6x+5)$	$\frac{-3}{4}x^2+\frac{11}{8}x+\frac{5}{3}$
−28	7	4	2	−3	$\frac{-1}{28}(7x+4)(2x-3)$	$\frac{-1}{2}x^2+\frac{13}{28}x+\frac{3}{7}$
56	−2	7	8	11	$\frac{1}{56}(-2x+7)(8x+11)$	$\frac{-2}{7}x^2+\frac{17}{28}x+\frac{11}{8}$

$$A=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Figure 2

$$A=\frac{1}{2}\begin{vmatrix} (x_1-x_3) & (y_1-y_3) \\ (x_2-x_4) & (y_2-y_4) \end{vmatrix}$$

Figure 3

which will bring about the desired result. For example, let the y-co-ordinates be -3, 5, 7, and 2; and let x_2 and x_4 be 8 and -5. Then the area formula becomes

$$A=\frac{1}{2}\begin{vmatrix} (x_1-x_3) & -10 \\ 13 & 3 \end{vmatrix}$$

$$=\frac{1}{2}\left[3(x_1-x_3)+130\right]$$

$$=\frac{3}{2}(x_1-x_3)+65.$$

The area will now be integral if (x_1-x_3) is an even integer, but fractional or a fraction with an odd number for numerator if (x_1-x_3) is an odd integer, or irrational if x_1 or x_3 is irrational.

Situation 5. An infinite sequence is not well defined unless there is given a general rule for finding any term. For example, there are many sequences beginning with 2, 4, 6. To convince students of this, you may exhibit several distinct sequences whose first few terms are alike. How can such sequences be found?

A solution. An infinite sequence may be considered as a function whose domain is the set of counting numbers $\{1, 2, 3, \cdots\}$. Each counting number is paired with exactly one member of the range of the function. Suppose we are interested in infinite sequences beginning 2, 4, 6. That means we are interested in functions (with the counting numbers as domain) which contain the ordered pairs (1, 2), (2, 4), and (3, 6). (See Fig. 4.) The *Lagrange interpolation formula* enables one to find polynomial expressions which will serve as

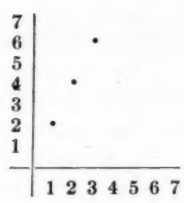


Figure 4

nth terms in such sequences. We proceed as follows.

Our aim is to construct a function F of the form

$$F(x)=y_1\cdot f_1(x)+y_2\cdot f_2(x)+y_3\cdot f_3(x)+y_4\cdot f_4(x).$$

Here y_1, y_2, and y_3 are the y-co-ordinates of the three points in Figure 4. We choose *any* point with an x-co-ordinate of 4 as our fourth point, for example, (4, 10). Its y-co-ordinate is used as y_4. Our problem now is to find some subsidiary functions f_1, f_2, f_3, and f_4 so that F will include all the four points already selected. Since we want $F(x_1)$ to equal y_1, we must arrange for $f_1(x_1)$ to equal 1, and the other three terms to equal zero. Since we want $F(x_2)$ to equal y_2, we must arrange for $f_2(x_2)$ to equal 1, and the other three terms to equal zero, and so on. If we let

$$f_1(x)=\frac{(x-2)(x-3)(x-4)}{-6},$$

then $f_1(1)=1$; and $f_1(2)$, $f_1(3)$, and $f_1(4)$ are all zero. Here the denominator equals the numerator evaluated for $x=1$.

The other subsidiary functions are constructed in an analogous way. Hence our F for this case may be defined as follows:

$$F(x)=\frac{2(x-2)(x-3)(x-4)}{-6}+\frac{4(x-1)(x-3)(x-4)}{2}+\frac{6(x-1)(x-2)(x-4)}{-2}+\frac{10(x-1)(x-2)(x-3)}{6}.$$

After simplifying we have

$$F(x)=\frac{1}{3}x^3-2x^2+\frac{17}{3}x-2.$$

Using n instead of x, we have thus the sequence

$$2, 4, 6, 10, \cdots, \left(\frac{1}{3}n^3-2n^2+\frac{17}{3}n-2\right), \cdots$$

If we had used (4, 8) as our fourth point, we would have obtained the sequence

$$2, 4, 6, 8, \cdots, 2n, \cdots,$$

and the point (4, 5) will determine the sequence

$$2, 4, 6, 5, \cdots, \left(\frac{-1}{2}n^3+3n^2-\frac{7}{2}n+3\right), \cdots.$$

Other choices for y_4 will yield other sequences beginning 2, 4, 6. To find several sequences in which the first *four* terms are the same, one may proceed in the same fashion, but with $F(x)$ equal to the sum of *five* terms.

Reviews and evaluations

(*Continued from p. 372*)

systems. This thorough coverage makes the chapter an excellent resource for teachers. Since the study of various systems of numeration is mainly to provide students with new insights into the decimal system, along with a bit of information about computers, one wonders if such thorough coverage of so many systems is necessary for average students. As for the superior student, instead of receiving the ready-made chart comparing systems, it might be well if he undertook to make one of his own for the class. In this unit, too, you will enjoy learning, along with your class, the names of some of the newer units of measure mentioned during the study of scientific notation. Do you know what a parsec is? Or a gigadollar?

The concept of set with its symbolism is introduced in Unit Two. The purpose of such a unit for younger pupils is to develop readiness for a future program of contemporary mathematics in senior high school, but it is remarkable to see the amount of suitable information that can be offered in the twenty-five pages that Stein allots to open sentences, solutions, solution sets, solutions of inequalities, subsets, operations with sets, Venn diagrams, and a glimpse at the symbols and truth tables of logic.

Unit Three is about graphing sentences in one and two variables. After a brief discussion of ordered pairs or numbers, absolute value, domain, and range, there follows an explanation of the graphing of equations and inequalities in one variable on a number line. Stein's placement of graphing on a number line in this final unit makes for a logical, well-rounded development of graphs; however, it deprives one of the use of this device to increase the understanding of sets. We suggest using graphs on number lines early in Unit Two. The rest of Unit Three is devoted to graphing equations and inequalities in two variables, both on a restricted number plane (lattice) and on the real number plane. The illustrations of graphs are excellent.

The supplementary units might have included an exercise on functions as sets of ordered pairs, and they might have included a lesson on zero or one on operations with powers other than ten. But since a supplement must contain only selected units, and since the author has selected wisely according to recent recommendations for the grade levels for which he is writing, we encourage your consideration of the supplement's possibilities.—*Lottchen Lipp Hunter, Wichita High School West, Wichita, Kansas.*

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Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D.C.

NCTM convention dates

JOINT MEETING WITH NEA

June 28, 1961
Atlantic City, New Jersey
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D.C.

TWENTY-FIRST SUMMER MEETING

August 21–23, 1961
University of Toronto, Toronto, Canada
Father John C. Egsgard, C.S.B., St. Michael's College School, 1515 Bathurst Street, Toronto 10, Canada

FORTIETH ANNUAL MEETING

April 15–18, 1962
Jack Tar Hotel, San Francisco, California
Kenneth C. Skeen, 3355 Cowell Road, Concord, California

Other professional dates

Men's Mathematics Club of Chicago and Metropolitan Area

May 19, 1961
YMCA Hotel, 826 South Wabash Avenue, Chicago, Illinois
Vernon R. Kent, 1510 South Sixth Avenue, Maywood, Illinois

Twenty-first Summer Meeting

University of Toronto,
Toronto, Ontario, Canada
August 20, 21, 22, 23, 1961

Host Organization:
Ontario Association of Teachers of Mathematics and Physics

Program Committee:
Clarence Ethel Hardgrove, Clifford Bell, Mildred Cole, Eunice Lewis

Local Chairman:
Father John C. Egsgard, C.S.B., St. Michael's College School, Toronto

PROGRAM HIGH LIGHTS OF GENERAL SESSIONS

Banquet:
"Education in the 1960's"
Honorable John Robarts, Ontario Minister of Education

Others:
"Principles of Learning and the Learning of Mathematical Principles"
Delox Wickens, Professor of Psychology, The Ohio State University
"Advanced Mathematics from an Elementary Standpoint"
Ralph D. James, Professor of Mathematics, University of British Columbia
"On the Nature of Applied Mathematics"
Henry O. Pollak, Bell Telephone Laboratories, Inc.

LECTURES

Nine lectures are scheduled for the purpose of improving or refreshing the mathematics background of teachers. Some of these are for secondary and college teachers, and these are for junior high school and elementary teachers.

A two-session short course on the topic "Different Approaches to Exponential and Trigonometric Functions" will be given by Donald E. Richmond of Williams College and a three-session short course on "Solution of Simultaneous Equations by Use of Matrices," by E. F. Beckenback of University of California, Los Angeles. Richard D. Anderson, Louisiana State University, will lecture on a topic related to geometry.

The junior high school and elementary teachers may hear a series of two lectures by Stanley Jackson, University of Maryland, on elementary geometry and a lecture by Burton H. Calvin, Boeing Scientific Research Laboratories, on a phase of arithmetic.

SECTION MEETINGS

A series of programs are being sponsored by the Ontario Association of Teachers of Mathematics and Physics. Speakers will describe the work of the Ontario Mathematics Commission and of experimental programs. Mr. Leonard Casiats and Mr. Hugh Fullerton of KCS Limited will discuss electronic computers and their use in business.

Affiliated groups from New England, New York, Ohio, and Pennsylvania are sponsoring sectional meetings.

There will be a session on "Teaching Machines."

Professor E. G. Begle, Director, School Mathematics Study Group, and Professor Richard Alpert, Princeton, will report on "An Evaluation of Junior and Senior High School SMSG Materials" and "Attitudes toward Mathematics."

State representatives

It has become almost a tradition to print in a spring issue of THE MATHEMATICS TEACHER a list of the state representatives of the Council. During a long period of years, our state representatives as a group have rendered invaluable service to our program. They have used a variety of methods, including personal solicitation, announcements at meetings, and sending reminders to "expirations," to promote membership in the Council. They have advertised our publications, even to the extent in some cases of operating actual sales programs at meetings. They have announced Council meetings and activities. In many cases, they have been able to gather useful information and news from the field. Below is the list of representatives who are serving during the 1960–61 school year.

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More about 1960-1961 Committees

The following omissions were made in giving the list of names of persons appointed to NCTM 1960–1961 Committees and Representatives (February, 1961 issue). Kindly regard these as a part of the earlier report.

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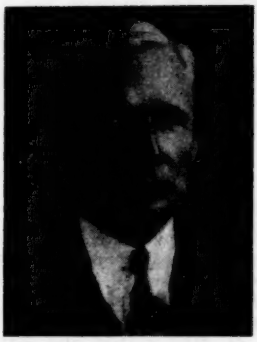
William David Reeve



All of us who knew him, as a colleague working in the interest of mathematics and as a friend, mourn the death of Dr. W. D. Reeve on February 16, 1961. Dr. Reeve was Professor Emeritus of Mathematics and former head of the Department of the Teaching of Mathematics at Teachers College, Columbia University.

Dr. Reeve was born in Edwardsport, Indiana, on September 11, 1883. After attending the local public schools, he was graduated from the Indiana State Normal School at Terre Haute in 1907. Feeling the need for further training and having heard of the great faculty at the University of Chicago gathered by President William Rainey Harper—Bolza, Slaught, J. W. A. Young, Milliken, Judd, and Gale, to mention a few—he studied there for a Bachelor of Science degree, which was granted in 1909. After a year of graduate work at the University of Chicago and a move to the University High School, Dr. Reeve completed the work for the Doctor of Philosophy degree at the University of Minnesota. It was awarded in 1924.

Dr. Reeve began his teaching career in the rural schools of Indiana. Like so many young men of his generation, he taught in a one-room country school immediately after graduation from high school in order to accumulate funds for continuing his education. He served as teacher and principal in his home state for several years and from 1910 to 1914 taught mathematics at the University High School in Chicago. There he came under the influence of Franklin W. Johnson and Ernest R. Breslich. He was head of the Department of Mathematics at University High School in Minnesota and also served as principal.

In 1923 he was appointed as Associate in Mathematics at Teachers College, Columbia University. Dr. Reeve became Professor of Mathematics at the college in 1927 and head of the Department of the Teaching of Mathematics in 1928. He retired in 1948 as Professor Emeritus.

Professor Reeve was one of the leaders in the founding of the National Council of Teachers of Mathematics and the editor of THE MATHEMATICS TEACHER, the official journal of the organization, from 1928 to 1950. It was he who urged initiating a series of publications of the National Council, the Yearbooks. He was the editor of numbers four to twenty for 1929 through 1948.

In addition to being a member of the National Council and the Mathematical Association of America, Professor Reeve was a life member of the National Education Association and the Central Association of Science and Mathematics Teachers. He also was an honorary member of the Mathematical Association of Great Britain and was awarded an honorary key by Phi Delta Kappa. On March 31, 1953, the Regents of the University of Minnesota, upon unanimous recommendation of the Faculty Committee on Honors and the Administrative Committee of the Senate, presented him with the Outstanding Achievement Award of the University, an award for former students "who have achieved high eminence and distinction."

Dr. Reeve was a member of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics whose report *The Place of Mathematics in Secondary Education* was published as the Fifteenth Yearbook. He was the author, or co-author, of sixteen textbooks. The one he considered his major work was *Mathe-*

matics for the Secondary School—Its Content and Methods of Teaching and Learning, published in 1954.

It is impossible to evaluate the influence and effect of a man in a given field. One can say, though, that Dr. Reeve left his mark on students and field alike; he was a leader in mathematics education. Some will remember him as an inspiring teacher, others as a man of firm convictions; all will remember him as a warm and charming host.—MYRON F. ROSSKOPF

Elizabeth Marie Gugle

In the evening hours of April 3, 1960, Miss E. Marie Gugle, a charter member of the National Council of Teachers of Mathematics and its fourth president, died at her home in Columbus, Ohio, at the age of 83. During her long professional career she made a great contribution to mathematics education, and every member of the National Council of Teachers of Mathematics, past or present, has profited from the wisdom and judgment of this distinguished lady.

Miss Gugle's leadership in education was widely recognized, and her wholesome influence rapidly spread beyond the borders of Columbus. She has a distinguished record as lecturer, educational consultant, and author. Her influential pen is responsible for numerous articles, and in 1920 she published a three volume series of mathematics texts entitled *Modern Junior Mathematics*. Early in her professional career she recognized that she could more effectively serve the interests of mathematics education through active association with organizations devoted to the study of educational problems. She joined the Central Association of Science and Mathematics Teachers in 1907 and in 1916 became the first woman to serve as president of this organization. She was an active member of the Department of Superintendence of the National Education Association and was elected to a two-year term as secretary of this department. She was especially proud of the fact that she was a charter member of the National Council of Teachers of Mathematics and served as its fourth president, 1926–28, the first woman to hold this office. She also served three terms as a member of the Board of Directors, and today we are the beneficiaries of her constructive and unselfish service.

In a series of notes on the early history of the Council, one of the former presidents writes that "the records will not reveal all of the consecrated work that Marie Gugle did for the Council." He is undoubtedly correct, for records speak of motions and resolutions and tell of tangible action. As president and director of the NCTM, Miss Gugle reflected a capacity for leadership which inspired those whom she so effectively served. But how can one record those intangible qualities which command the respect of people and stir them to action? Any professional organization is the personification of those who create it, and while today Miss Gugle's voice is silent, she continues to speak to an ever widening audience.—HAROLD P. FAWCETT

Joint meeting with the NEA

With the co-operation of the Association of Mathematics Teachers of New Jersey, the NCTM will hold a meeting during the NEA Convention in Atlantic City, New Jersey, June 28, 1961. The meeting will open with a luncheon at noon at which John J. Kinsella, professor and chairman of mathematics education, New York University, will speak on the topic "What IS the Deadwood in the Mathematics Curriculum?"

At the General Session to follow at 2:00 p.m. there will be a panel discussion on "Experiences in Teaching Contemporary Mathematics." The panel will be composed of Florence Elder, chairman of the Mathematics Department, Junior-Senior High School, West Hempstead, New York and Gail Koplin, chairman of the Mathematics Department, Watchung Hills Regional High School, Plainfield, New Jersey. Following the panel discussion, Sheldon S. Myers, head of the Mathematics Section of the Test Development Division, Educational Testing Service, Princeton, New Jersey, will speak on the topic "Testing Programs and Contemporary Mathematics."

Harold Gouss and Bruce E. Meserve, president and president-elect respectively of the Association of Mathematics Teachers of New Jersey, will serve as presiding officers at the luncheon and General Session. The program has been arranged by a committee composed of Florence W. Borgeson, Evan M. Maletsky, Mary C. Rogers, and John A. Schumaker, chairman.

Those who wish to attend the luncheon should send reservations, with payment of $3.00 for each person, before June 10 to Mrs. Florence W. Borgeson, 307 Prospect Street, Westfield, New Jersey. The meeting will be held in the Shelburne Hotel.

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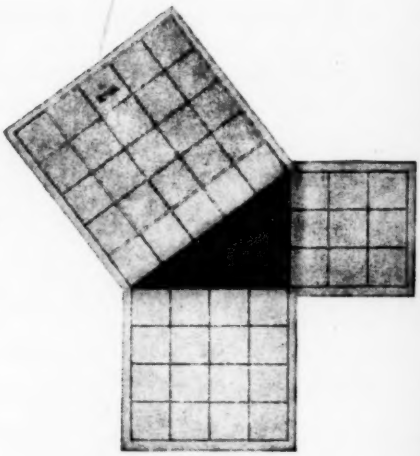

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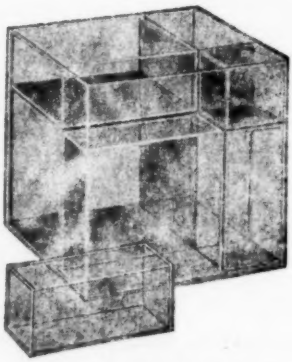

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